Note on Stratified vector bundle (str. v.b.)

Ref: https://arxiv.org/abs/2303.04200

In the following note I always use the word "good" which means a lot of conditions to exclude exotic cases. The explicite conditions can be found in reference.

Stratified space

[Def] A stratified space is a pair (X, Σ) where X is a "good" topo space and Σ is a "good" partition into locally closed sm mf. fulfilling frontier condition.

[Rmk] Frontier condition: If $\begin{cases} S, R \in \Sigma \\ S \cap \bar{R} \neq \phi \end{cases}$ then $S \subset \bar{R}$

[Exp] A CW complex partitioned into n-cells is a stratified space.

[Exp] Let $G \times M \to M$ be a "good" action.

1. $S_G(M) = \{\text{orbits of the action}\}$. Then $(M, S_G(M))$ is a stratified space.

2. $S_G(M/G) := \{ S/G \mid S \in S_G(M) \}$, then $(M/G, S_G(M/G))$ is a stratified space.

[Exp] Introduce new equivalence class to obtain a partition.

Lie groupoid $G \overset{s}{\underset{t}{\rightrightarrows}} M$ is for any $g \in G$, we have

$s(g) \xrightarrow{g} t(g)$ and satisfying some conditions.

(Similar to category)

Let $G_x = \{ g \in G \mid s(g) = t(g) = x \} = s^{-1}(x) \cap t^{-1}(x)$.

Let orbit $\mathcal{O}_x := t\, s^{-1}(x)$ and the normal space $N_x := T_x M / T_x \mathcal{O}_x$

We define a relation: $x \sim y \Leftrightarrow G_x \overset{\phi}{\underset{\sim}{\to}} G_y$, $N_x \overset{\psi}{\underset{\sim}{\to}} N_y$

and
$$\begin{tikzcd} G_x \times N_x \arrow[r, "\phi \times \psi"] \arrow[d, "\text{action}"'] & G_y \times N_y \arrow[d, "\text{action}"] \\ N_x \arrow[r, "\psi"] & N_y \end{tikzcd}$$

It's an equivalence class and we call this the stratification by Morita type.

Morphism

[Def] Let (X_1, Σ_1) and (X_2, Σ_2) be stratified space. A stra. mor. is a conti map $f: X_1 \to X_2$ s.t. $\forall S_1 \in \Sigma_1$, $\exists S_2 \in \Sigma_2$ ·

(1) $f(S_1) \subset S_2$

(2) $f|_{S_1}: S_1 \to S_2$ is sm.

[Exp] $G \rightrightarrows M$ be a "good" Lie groupoid. Then we have Morita type stratification $(M, S^m_G(M))$ and canonical stratification $(M/G, S_G(M/G))$. Then $\pi: M \to M/G$ is a stratified mor.

Smooth structure for stratified space and morphism

[Def] A sm structure on (X, Σ) is a choice of maximal atlas consisting of compatitive charts.

- A chart is a pair $(U, \varphi: U \to \mathbb{R}^n)$, where $U \overset{\text{open}}{\subseteq} X$, $n \in \mathbb{N}$, and φ is a "good" map.

  [Rmk] Different chart can have different "n". That's different from sm structure on m.f..

  [Rmk] What does "good" mean? φ is a locally closed embedding and for $\forall S \in \Sigma$, $\varphi(S \cap U)$ is an embedded subm.f. of $\mathbb{R}^n$.

- Compatitive charts: Let $\varphi: U \to \mathbb{R}^n$ and $\psi: V \to \mathbb{R}^m$ be two charts.

1. $n=m$. Then two charts are called compatible if $\forall p \in U \cap V$, $\exists p \in W \overset{\text{open}}{\subseteq} U \cap V$ and open n.b.h. O_U, O_V of $\varphi(W)$ and $\psi(W)$ s.t.

$$\begin{tikzcd}
O_U \arrow[rr, "\exists H", "\sim"'] & & O_V \\
\varphi(W) \arrow[u, "i"] \arrow[rr, phantom, "\circlearrowright"] & & \psi(W) \arrow[u, "i"'] \\
& W \arrow[ul, "\varphi"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

H: diffeomorphism.

2. $n \neq m$. Say $m > n$. Let $\iota: \mathbb{R}^n \to \mathbb{R}^m$.

We say $\varphi: U \to \mathbb{R}^n$ and $\psi: V \to \mathbb{R}^m$ are compatible if $\iota\varphi: U \to \mathbb{R}^m$ and $\psi: V \to \mathbb{R}^m$ are compatible.

A stratified space together with a sm structure will be called a differentiable stratified space.

Then we consider what's sm. mor.?

[Def] X, Y are differentiable stratified space.

$f: X \to \mathbb{R}$ is sm if for any chart $\varphi: U \to \mathbb{R}^n$, $\exists$ sm $g: \mathbb{R}^n \to \mathbb{R}$

s.t.
$$\begin{tikzcd}
U \arrow[r, "\varphi"] \arrow[dr, "f"'] \arrow[dr, phantom, shift left=3, "\circlearrowright"] & \mathbb{R}^n \arrow[d, "g"] \\
& \mathbb{R}
\end{tikzcd}$$

$F: X \to Y$ is said to be sm if for all $f: Y \to \mathbb{R}$, $f \circ F: X \to \mathbb{R}$ is sm.

[Rmk] Note that in sm mf we say $F: X \to Y$ sm if for all chart
$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "g"] & \mathbb{R}^m \\
U \arrow[u, "s"] \arrow[r, "f"'] \arrow[ur, phantom, "\circlearrowright"] & V \arrow[u, "\iota"']
\end{tikzcd}$$
g is sm. But here we can not use that because in stratified space, chart $\varphi: U \to \mathbb{R}^n$ is not a homeo but only an embedding as a locally closed subspace.

Sm structure tells you which function is sm function.
Interestingly, some "not sm" function may be sm!

[Exp] $(\mathbb{R}, \Sigma)$, where $\Sigma = \{\mathbb{R}_{<0}, \{0\}, \mathbb{R}_{>0}\}$ is a partition, is a stratified space.
Define $\phi : \mathbb{R} \to \mathbb{R}^2$, $x \mapsto (x, |x|)$ is a chart (globle chart).
So $(\mathbb{R}, \Sigma)$ is a differentiable stratified space.
We'll show $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$ is a sm function.

$\exists$ projection π_2 s.t.

$$\begin{tikzcd}
x & \mathbb{R} \arrow[r, "\phi"] \arrow[dr, "f"'] & \mathbb{R}^2 \arrow[d, "\pi_2"] \arrow[dl, phantom, "\circlearrowleft"] & (x,|x|) \\
& & \mathbb{R} & |x|
\end{tikzcd}$$

[Exp] Canonical stratification $(M, S_G(M))$ and $(M/G, S_G(M/G))$ are differentiable stratified space.

[Exp] Let M be a sm m.f. and Σ a stratification of M into embedded m.f.. Then (M, Σ) is a stratified space. Atlas of sm mf can be checked also atlas for stratified space. So (M, Σ) is a differentiable stratified space.

In general, differentiable stratified space is quite wild and we should impose various conditions: Whitney A, B, C.

Thm: If $G \rightrightarrows M$ is a proper Lie groupoid, then M equipped with its Morita type stratification and M/G with its canonical stratification $S_G(M/G)$ are both Whitney B stratified space.

Stratified vector bundles

[Def] A stratified vector bundle is a stratified mor $p : A \to X$ between stratified space (A, Σ_A) and (X, Σ_X) satisfying:

(1) $\forall S \in \Sigma_X$, $p^{-1}(S) \in \Sigma_A$

(2) $\forall S \in \Sigma_X$, $p: A|_S := p^{-1}(S) \longrightarrow X$ is a sm v.b.

(3) The scalar multiplication $\mu: \mathbb{R} \times A \to A$ is a stratified mor.

If A, X have sm structure, p, μ are sm maps, then we call $p: A \to X$ is differentiable.

[Exp] Let (X, Σ_X) be a stratified space.

Let $A = X \times \mathbb{R}^n$ and $\Sigma_A = \{ S \times \mathbb{R}^n \mid S \in \Sigma_X \}$.

Then the projection $p: A \to X$ is a stratified v.b. and we call it trivial vecter bundle.

$\boxed{\text{框架:}}$

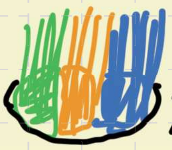
Stratifed v.b.

1. stratified str, bundle, sm charts 等结构的相容性.
2. 微分流形的推广 (研究的内容与 sm mf 基本相同)

$$\begin{tikzcd}
\text{stratified space and mors between them} \arrow[r, "+\text{ bundle}"] \arrow[d, "+\text{ sm structure (charts)}"] & \text{stratified vector bundle} \\
\text{sm stratified space and mors between them} \arrow[r, "+\text{ bundle}"] & \text{differentiable vector bundle}
\end{tikzcd}$$

stratified space 由 sm m.f. 拼成

stratified vector bundle 由 sm vector bundle 拼成.



X $\}\ A$