Sheaf theory   Ref: Gtm 65.

Big picture: Sheaf theory is a method to obtain global information from local information.

Motivation: Most problems can be solved without sheaf theory. But without sheaf theory makes things hard to comprehence.

presheaves and sheaves

[Def] A presheaf $\mathcal{F}$ over a topological space X is

(a) An assignment to each nonempty open set $U \subset X$ of a set $\mathcal{F}(U)$ with elements called sections.

(b) A collection of mappings (called restriction homomorphisms)

$$r^U_V : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

for each pair of open sets U and V s.t. $V \subset U$ satisfying

(1) $r^U_U = id_U$ (2) For $U \supset V \supset W$, $r^U_W = r^V_W \circ r^U_V$.

[Def] (mor. of presheaves) Let $\mathcal{F}, \mathcal{G}$ be two presheaves over X.

A morphism $h: \mathcal{F} \to \mathcal{G}$ is a collection of maps

$$h_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

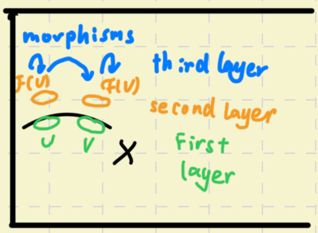
for each open set U in X s.t. the following diagram commutes

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, "r^U_V"'] & \mathcal{G}(U) \arrow[d, "r^U_V"] \\ \mathcal{F}(V) \arrow[r] & \mathcal{G}(V) \end{tikzcd} \qquad V \subset U \subset X$$

$\mathcal{F}$ is said to be a subpresheaf of $\mathcal{G}$ if the maps h_U above are inclusions.

[Rmk] Roughly speaking, presheaf over X has three Layers.

third layer	$\mathrm{Hom}(\mathcal{F}(U), \mathcal{F}(V))$	Hom sets between $\mathcal{F}(U)$ and $\mathcal{F}(V)$
Second layer	$\mathcal{F}(U)$ $\mathcal{F}(V)$	each open set assign a set $\mathcal{F}(\cdot)$
First layer	U V	open sets in X



$$\mathrm{Hom}(\mathcal{F}(U), \mathcal{F}(V)) = \begin{cases} id_U & U = V \\ r^U_V & U \supseteq V \\ \phi & o/w \end{cases}$$

When $U \subseteq V$ and we consider sheaf of functions, $\mathrm{Hom}(\mathcal{F}(U), \mathcal{F}(V))$ contains inclusions.

Then mors of presheaves should preserve this 3 layers.

$\mathcal{F}, \mathcal{G}$ be two presheaves over X. A mor $h: \mathcal{F} \to \mathcal{G}$ is assign each element an element in the same layer compatitively.

third layer	$\mathrm{Hom}(\mathcal{F}(U), \mathcal{F}(V)) \to \mathrm{Hom}(\mathcal{G}(U), \mathcal{G}(V))$, with $\{r^U_{\mathcal{F}V}\}$ and $\{r^U_{\mathcal{G}V}\}$
second layer	$h_U: \mathcal{F}(U) \longrightarrow \mathcal{G}(V)$
First layer	$U \longmapsto V$

Can be simplified to be just a family of $h_U: \mathcal{F}(U) \to \mathcal{F}(V)$ because the assignment at first and third layer is fixed.

$h: \mathcal{F} \to \mathcal{G}$ are those maps satisfying:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "h_U"] \arrow[d, "r^U_{\mathcal{F}V}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{G}(U) \arrow[d, "r^U_{\mathcal{G}V}"] \\
\mathcal{F}(V) \arrow[r, "h_V"] & \mathcal{G}(V)
\end{tikzcd}$$

*Actually, I belive presheaf over X is a 2-cat and mors are 2-functors. (to check it's a 2-cat is so awful and seems not very useful at this stage, so it's just a guess. But it's easy to prove the second and third layer combine satisfying conditions to form a 1-cat).

I think this "category version" or just "layer version can explicity show what data presheaves contain.

[Rmk] When we endow more structure to $\mathcal{F}(U)$, e.g. $\mathcal{F}(U)$ is a group, all mors in def should be grp homo.

[Def] A presheaf $\mathcal{F}$ is called a sheaf if for every collection U_i of open subsets of X with $U = \bigcup U_i$ then $\mathcal{F}$ satisfies

- Axiom S_1: If $s, t \in \mathcal{F}(U)$ with $r^U_{U_i}(s) = r^U_{U_i}(t)$ then $s = t$.
- Axiom S_2: If $s_i \in \mathcal{F}(U_i)$ and for $U_i \cap U_j \neq \emptyset$ we have

$$r^{U_i}_{U_i \cap U_j}(s_i) = r^{U_j}_{U_i \cap U_j}(s_j), \text{ for } \forall i, j$$

then there exists an $s \in \mathcal{F}(U)$ s.t. $r^U_{U_i}(s) = s_i$ for $\forall i$.

[Rmk] For "good" patches of local functions, we can glue them to a global one. Axiom S_2 convices existence and Axiom S_1 convinces uniqueness.

[Rmk] mors of sheaves are the same as mors of presheaves.

[Exp] (presheaf and not a sheaf) $X = \{a, b\}$ with discrete topo.
$\mathcal{F}(a) = \mathcal{F}(b) = \underbrace{\mathbb{k}}$ (field). and restrictions are all zero. Then it violates Axiom S_1.

Then what's the case on m.f.? What's presheaves on m.f.?

Idea:

M (manifold) ← S-structure → tells you what's S-functions on M.
↓
construct sheaves of S-functions. □

Let S = differentiable $\mathcal{E}$, real-analytic $\mathcal{A}$, or complex-analytic $\mathcal{O}$.
$\mathcal{E}$ ⇝ C^∞ functions, $\mathcal{A}$ ⇝ real-analytic functions, $\mathcal{O}$ ⇝ holomorphic functions

[Def] (S-structure) An S-structure S_M on a k-manifold M is a family of k-valued continous functions defined on the open sets of M s.t.

(1) $\forall p \in M$, $\exists$ open n.b.h. $U \ni p$ and a homeo $U \to U' \subseteq k^n$ s.t. $\forall$ open $V \subset U$, $f: V \to k \in S_M$ iff $f \circ h^{-1} : h(V) \to k \in S(h(V))$

(2) If $f: U \to k$ where $U = \bigcup_i V_i$ and U_i open in M, then $f \in S_M$ iff $f|_{U_i} \in S_M$. (e.g. $U = \bigcup_{p \in U} U_p$, U_p is open n.b.h. of p then we can use (1) in def)

(M, S_M) is a S-manifold.

[Def] $C_X(U) :=$ conti functions $X \to k$, it's a sheaf of X.

[Def] (Structure sheaf of the m.f.) Let X be a S-manifold.
$S_X(U) :=$ the S-functions on U. defines a subsheaf of C_X
$\mathcal{E}_X, \mathcal{A}_X, \mathcal{O}_X$ are sheaves of differentiable, real-analytic and holomorphic functions on a mf X.

[Rmk] One may think S-structure is just a sheaf. That's wrong. S-structure just tells you what's S-function on the m.f.. S-structure is an instruction book, then we call tell sheaf of S-functions on S-manifold M, which is so called sheaf structure.

Presheaf of modules occur very often in the world of m.f. We'll see tight relationship between sheaf of modules and $\mathcal{S}$-bundles.

[Def] $\mathcal{R}$ is a presheaf of commutative ring and $\mathcal{M}$ is a presheaf of abelian groups, both over a topo space X. We say $\mathcal{M}$ is a presheaf of $\mathcal{R}$-modules if

(1) For each open $U \subseteq X$, $\mathcal{M}(U)$ is a $\mathcal{R}(U)$-module.

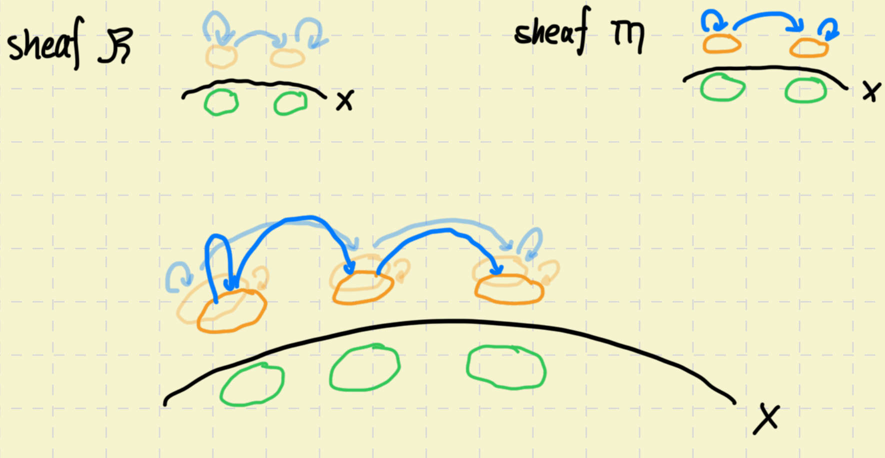
(2) For each $V \overset{\text{open}}{\subseteq} U \overset{\text{open}}{\subseteq} X$, $\forall \alpha \in \mathcal{R}(U)$

$$\begin{tikzcd} \mathcal{M}(U) \arrow[r, "\alpha\circ -"] \arrow[d, "r_{\mathcal{M}\,V}^{\;U}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{M}(U) \arrow[d, "r_{\mathcal{M}\,V}^{\;U}"] \\ \mathcal{M}(V) \arrow[r, "r_{\mathcal{R}\,V}^{\;U}(\alpha)\circ -"] & \mathcal{M}(V) \end{tikzcd}$$

(compatibleness of module structure and restriction in sheaf structure)

If $\mathcal{M}$ is a sheaf, then we say $\mathcal{M}$ is a sheaf of $\mathcal{R}$-modules.

[Rmk]



[Exp] Let $E \to X$ be an $\mathcal{S}$-bundle. Define a presheaf $\mathcal{S}(E)$ by setting $\mathcal{S}(E)(U) = \mathcal{S}(U, E)$, sections of E over U for $U \overset{\text{open}}{\subseteq} X$, together with natural restrictions. $\mathcal{S}(E)$ is called the sheaf of $\mathcal{S}$-sections of the vector bundle E. $\mathcal{S}(E)$ is a sheaf of $\mathcal{S}_X$-modules for an $\mathcal{S}$-bundle $E \to X$. For example, we have sheaves of differential forms $\mathcal{E}_X^*$ on a differentiable m.f., or the sheaf of differential forms of type (p, q), $\mathcal{E}_X^{p,q}$ on a complex m.f. X.

[Exp] Let $\mathcal{O}_{\mathbb{C}}$ denote the sheaf of holo functions in $\mathbb{C}$. Let $\mathcal{J}$ denote the sheaf by setting

$$\begin{cases} \mathcal{J}(U) = \mathcal{O}_{\mathbb{C}}(U) & \text{if } 0 \notin U \\ \mathcal{J}(U) = \{f \in \mathcal{O}_{\mathbb{C}}(U) \mid f(0) = 0\} & \text{if } 0 \in U \end{cases}$$

$\mathcal{J}$ is a sheaf of $\mathcal{O}_{\mathbb{C}}$-modules.

[Def] Let X be a complex m.f. with structure sheaf $\mathcal{O}_X$. Then a sheaf of $\mathcal{O}_X$-modules is called an analytic sheaf.

[Rmk] we introduce analytic sheaf because it ocurrs frequently.

The rest of this part we focus on the relationship between bundles and sheaves. Just as in algebraic geometry, we hope to find a correspondence between {bundles over X} and {sheaves over X}. Clearly, to make correspondence holds, we need put restrictions on bundles and sheaves, i.e., the question is to find "??" in the following and prove the bijection

$$\{?? \text{ bundles over } X\} \xleftrightarrow{1:1} \{?? \text{ sheaves over } X\}$$

[Def] Let $\mathcal{R}$ be a sheaf of commutative rings over a topological space X.

(a) Define $\mathcal{R}^p$, for $p \geq 0$, by setting $\mathcal{R}^p(U) = \overbrace{\mathcal{R}(U) \oplus \cdots \oplus \mathcal{R}(U)}^{p\text{-terms}}$ and natural restriction. $\mathcal{R}^p$ is a sheaf and we call $\mathcal{R}^p$ the direct sum of $\mathcal{R}$. ($p=0$ corresponding to 0-module)

(b) If $\mathcal{M}$ is a sheaf of $\mathcal{R}$-modules s.t. $\mathcal{M} \cong \mathcal{R}^p$ for some $p \geq 0$ then $\mathcal{M}$ is said to be a free sheaf of modules.

(c) If $\mathcal{M}$ is a sheaf of $\mathcal{R}$-modules s.t. each $x \in X$ has a n.b.h. U s.t. $\mathcal{M}|_U$ is free, then $\mathcal{M}$ is said to be locally free.

[Rmk] $\mathcal{M}|_U$ is the restriction of sheaf $\mathcal{M}$, the def can be guessed easily and we left as an exercise.

[Exp] Let $\mathcal{M}$ be the locally free sheaf of $\mathcal{S}$-module

where $\mathcal{S}$ is the structure sheaf of $\mathcal{S}$-manifold $(X, \mathcal{S})$.
Then for each $x \in X$, $\exists$ a n.b.h. U of x s.t. $\mathcal{M}|_U \cong (\mathcal{S}|_U)^r$.
To unwrap the equation, for each open $V \subseteq U$, we have
$\mathcal{M}|_U(V) \cong (\mathcal{S}|_U)^r(V)$, i.e., $\mathcal{M}(V) \cong \mathcal{S}(V)^r = \{(g_1, \cdots, g_r) \mid g_i \in \mathcal{S}(V)\}$

$= \left\{ f: V \to k^r \;\middle|\; \begin{array}{l} \text{write } f = (g_1, \cdots, g_r) \\ g_i \in \mathcal{S}(V) \end{array} \right\}$

Hence, locally free sheaf of $\mathcal{S}$-module means for each $x \in X$ there exists a n.b.h. U_x of x s.t. $\mathcal{M}(U)$ are vector-valued function with each component a $\mathcal{S}$-function.

[Thm] Let $X = (X, \mathcal{S})$ be a connected $\mathcal{S}$-m.f. There is a bijection

$$\{\text{iso classes of } \mathcal{S}\text{-bundles over } X\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{iso classes of locally free sheaves} \\ \text{of } \mathcal{S}\text{-modules over } X \end{array} \right\}$$

Pf: $\Rightarrow$ Given a $\mathcal{S}$-bundle $E \to X$, we need to construct a locally free sheaves of $\mathcal{S}$-modules over X where $\mathcal{S}$ is the structure sheaf.
We claim sheaf $\mathcal{S}(E)$ is the corresponding locally free sheaf of $\mathcal{S}$-modules. It suffices to show $\mathcal{S}(E)$ is locally free.
By local triviality of bundle E, for any $x \in X$ there exists a n.b.h. U of x, s.t. $E|_U \cong U \times k^r$. key: Pass this iso to sheaf.
Claim: $\mathcal{S}(E)|_U \cong \mathcal{S}(U \times k^r)$ Indeed, for $\forall V$ open in U, we have
$\mathcal{S}(E)|_U(V) = \mathcal{S}(E)(V) = \mathcal{S}(V, E) = \mathcal{S}(V, U \times k^r) = \mathcal{S}(U \times k^r)(V)$
Thus $\mathcal{S}(E)|_U = \mathcal{S}(U \times k^r)$.

Claim: $\mathcal{S}(U \times k^r) \cong \underbrace{\mathcal{S}|_U \oplus \cdots \oplus \mathcal{S}|_U}_{\# r}$

It suffices to show $\mathcal{S}(U \times k^r)(V) \cong \mathcal{S}|_U \oplus \cdots \oplus \mathcal{S}|_U(V)$ for any $V \overset{\text{open}}{\subseteq} U$.

$$\mathcal{S}(U \times k^r)(V) = \mathcal{S}(V, U \times k^r) = \left\{ \begin{array}{l} f: V \to V \times k^r \\ \quad x \mapsto (x, g(x)) \end{array} \;\middle|\; \begin{array}{l} g: V \to k^r, \text{ write as} \\ (g_1, \cdots, g_r), \text{ satisfying } g_i \in \mathcal{S}(V) \end{array} \right\}$$

$$\mathcal{S}(U\times k^r)(V) \longleftrightarrow \mathcal{S}|_U \oplus \cdots \oplus \mathcal{S}|_U(V) = \mathcal{S}(V)^r$$

$$f \longmapsto (g_1, \cdots, g_r) = g$$

$$\begin{array}{c} f: V \to V\times k^r \\ x \mapsto (x, g(x)) \end{array} \longleftarrow g$$

It's clearly an iso.

$\Leftarrow$ Given a locally free sheaf of $\mathcal{S}$-module $\mathcal{L}$, we w.t. construct a $\mathcal{S}$-bundle over X.

Since $\mathcal{L}$ is locally free, we can find an open covering $\{U_\alpha\}$ of X and a family of sheaf iso $g_\alpha : \mathcal{L}|_{U_\alpha} \xrightarrow{\sim} \mathcal{S}^r|_{U_\alpha}$

[Rmk] r doesn't depend on U_α since X is connected.

Define $g_{\alpha\beta} : \mathcal{S}^r|_{U_\alpha\cap U_\beta} \to \mathcal{S}^r|_{U_\alpha\cap U_\beta}$ by $g_{\alpha\beta} = g_\alpha g_\beta^{-1}$.

Since g_α, g_β are sheaf maps, $g_{\alpha\beta}$ is also a sheaf map.

Sheaf map $g_{\alpha\beta}$ is a family of mors, one of them is

$$(g_{\alpha\beta})_{U_\alpha\cap U_\beta} : \underset{\mathcal{S}(U_\alpha\cap U_\beta)^r}{\overset{}{\mathcal{S}^r|_{U_\alpha\cap U_\beta}(U_\alpha\cap U_\beta)}} \longrightarrow \underset{\mathcal{S}(U_\alpha\cap U_\beta)^r}{\mathcal{S}^r|_{U_\alpha\cap U_\beta}(U_\alpha\cap U_\beta)}$$

where $\mathcal{S}^r|_{U_\alpha\cap U_\beta}(U_\alpha\cap U_\beta) = \mathcal{S}(U_\alpha\cap U_\beta)^r$.

Claim: The sheaf map $g_{\alpha\beta}$ is equivalent to the map

$$g_{\alpha\beta} : U_\alpha\cap U_\beta \longrightarrow GL(r,k)$$

Indeed, $\mathcal{S}(U_\alpha\cap U_\beta)^r = \{(g_1,\dots,g_r) \mid g_i \in \mathcal{S}(U_\alpha\cap U_\beta)\}$ is a vector of functions. We can also view it as a vector-valued map.

$$\mathcal{S}(U_\alpha\cap U_\beta)^r = \{ f: \begin{array}{c} U_\alpha\cap U_\beta \to k^r \\ x \mapsto f(x) \end{array} \mid f(x) = (g_1(x),\dots,g_r(x)),\ g_i \in \mathcal{S}(U_\alpha\cap U_\beta)\}.$$

Hence, $$(g_{\alpha\beta})_{U_\alpha\cap U_\beta} : \mathcal{S}(U_\alpha\cap U_\beta)^r \longrightarrow \mathcal{S}(U_\alpha\cap U_\beta)^r$$
$$[f: U_\alpha\cap U_\beta \to k^r] \longmapsto [h: U_\alpha\cap U_\beta \to k^r]$$

i.e., $$(g_{\alpha\beta})_{U_\alpha\cap U_\beta} : U_\alpha\cap U_\beta \longrightarrow GL(r,k)$$
$$x \longmapsto g_{\alpha\beta}(x) \quad \text{s.t.} \quad h(x) = g_{\alpha\beta}(x) f(x)$$

Then $(g_{\alpha\beta})_V = (g_{\alpha\beta})_{U_\alpha\cap U_\beta}|_V$. So $\exists$ a map $g_{\alpha\beta} : U_\alpha\cap U_\beta \to GL(r,k)$ equivalent to the original sheaf map $g_{\alpha\beta}$.

Let $\tilde{E} = \bigcup_\alpha U_\alpha \times k^r / \sim$ where $\sim$ is $(x, \xi) \sim (x, g_{\alpha\beta}(x)\xi)$, $U_\alpha \cap U_\beta \neq \phi$.

The trivialization of $\tilde{E}$ is $[U_\alpha \times k^r] \xrightarrow{\sim} U_\alpha \times k^r$.

Since $g_{\alpha\beta} \circ g_{\beta\gamma} = g_\alpha g_\beta^{-1} g_\beta \circ g_\gamma^{-1} = g_\alpha g_\gamma^{-1} = g_{\alpha\gamma}$, $\{g_{\alpha\beta}\}$ are transition functions for vector bundle $\tilde{E}$.

The correspondence doesn't depend on representation of iso classes. Then let's check it's a bijection.

$E \mapsto S(E) \mapsto \tilde{E} = \bigcup U_\alpha \times k^r / \sim$, $(x, \xi) \sim (x, g_{\alpha\beta}\xi)$ where U_α is the triviality of sheaf $S(E)$

By construction, U_α is also the triviality of bundle E. Hence they're the same.

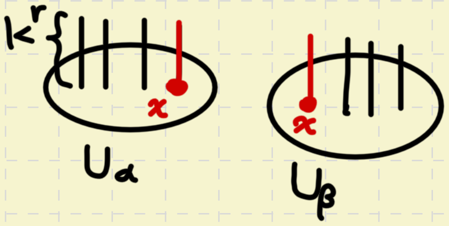
$S(E) \mapsto \tilde{E} \mapsto S(\tilde{E})$

trivialization on U_α which is also trivialization of $S(\tilde{E})$.

[Rmk] How bundles and locally free sheaf of S-module related?

We only consider construction of a bundle from the sheaf.

To construct a bundle, we need to glue $\{U_\alpha \times k^r\}_\alpha$, i.e., let $E = \coprod U_\alpha \times k^r / \sim$. So we only need to consider how to glue, i.e., what's equivalence relation '$\sim$'? The following picture shows that to glue two trivialization $U_\alpha \times k^r$ and $U_\beta \times k^r$, we only need to assign each $x \in U_\alpha \cap U_\beta$ an element in $G(r, k)$, which is an automorphism on k^r.



for $x \in U_\alpha \cap U_\beta$, it suffices to glue two fibers k^r to a fiber. It's equivalent to give an iso $\begin{aligned} k^r &\to k^r \\ \xi &\mapsto g_{\alpha\beta}\xi \end{aligned}$, then we can glue two fibers by $(x, \xi) \sim (x, g_{\alpha\beta}\xi)$.

$g_{\alpha\beta}: U_\alpha \cap U_\beta \to GL(r, k)$ exactly plays this role. $\square$

We'll end this part by introduce the generalization of locall free sheaves. This generation can even be defined on complex m.f. with singularities — complex spaces. An analytic sheaf on a complex mf. X is said to be coherent if for each $x \in X$ there is a n.b.h. U of x s.t. there is an exact sequence of sheaves over U, $\mathcal{O}^p|_U \to \mathcal{O}^q|_U \to \mathcal{F}|_U \to 0$ for some p and q. More detailed can be see in Gathmann's algebraic geometry.

Resolutions of sheaves

Motivation:

A sheaf on X is a carrier of localized information about the space X. To get global information, we need to apply homological alg to sheaves. In this section we'll do the prework.

[Def] An étalé space over a topo space X is a topo space Y together with a continous surj mapping $\pi: Y \to X$ s.t. π is a local homeo.

[Exp] (Relationship between bundles) Let $\pi: E \to X$ be a bundle over X. Then surj map $\pi: E \to X$ locally is $\pi|_U : U \times k^r \to U$ is a homeo since k^r is contractible.

From the example, étalé space is a generalization of bundles. So we can also define sections for étalé space.

[Def] A section of an étalé space $Y \xrightarrow{\pi} X$ over an open set $U \subseteq X$ is a continous map $f: U \to Y$ s.t. $\pi \circ f = id_U$. The set of sections over U is denoted by $\Gamma(U, Y)$.

Question: Given a presheaf $\mathcal{F}$ over X, can we construct an étalé space $\widetilde{\mathcal{F}} \to X$ associated to $\mathcal{F}$? The answer is yes and we have:

[Slogan] étalé space associated to presheaf is the union of stalks.

[Def] (stalk) Let $\mathcal{F}$ be a presheaf over X. Let $\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U)$ w.r.t. restriction maps $\{r^U_V\}$. We call $\mathcal{F}_x$ the stalk of $\mathcal{F}$ at x

[Rmk] The direct sum $\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U)$ means there are $\{\mathcal{F}_x, r^U_x \mid U \ni x\}$,

s.t.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "r^U_V"] \arrow[dr, "r^U_x"'] & \mathcal{F}(V) \arrow[d, "r^V_x"] \\ & \mathcal{F}_x \end{tikzcd}$$

for any $x \in U, V$ and for each commutative

(f_U, f_V are datas of $\varinjlim$)

diagram $\begin{tikzcd} \mathcal{F}(U) \arrow[rr, "r^U_V"] \arrow[dr, "h_U"'] & & \mathcal{F}(V) \arrow[dl, "h_V"] \\ & W & \end{tikzcd}$, there exists unique $g: \mathcal{F}_x \to W$

s.t. the new diagram commutes

$$\begin{tikzcd} \mathcal{F}(U) \arrow[rr, "r^U_V"] \arrow[d, "r^U_x"'] \arrow[drr, "h_U" near start] & & \mathcal{F}(V) \arrow[d, "h_V"] \arrow[dll, "r^V_x"' near start] \\ \mathcal{F}_x \arrow[rr, dashed, "\exists !"] & & W \end{tikzcd}$$

[Rmk] If the structures are preserved by direct sum $\varinjlim_{x \in U}$, then $\mathcal{F}_x$ inherent this structure. For instance, if $\mathcal{F}(U)$ is abelian group or commutative ring, then so is $\mathcal{F}_x$ for $x \in U$.

[Def] Consider data of the direct sum $r^U_x : \mathcal{F}(U) \to \mathcal{F}_x$. If $s \in \mathcal{F}(U)$, we call $s_x := r^U_x(s)$ the germ of s at x and s is called a representative for the germ s_x.

[Rmk] Presheaf v.s. Stalk v.s. Germ.
$\mathcal{F}$ $\quad$ $\mathcal{F}_x$ $\quad$ s_x

presheaf valued at U

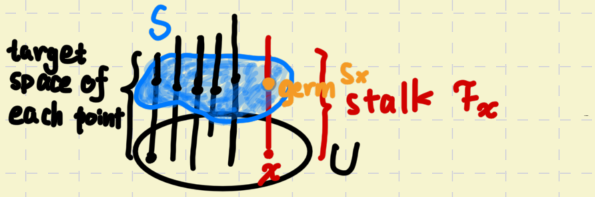
$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}_x \\ s & \longmapsto & s_x \end{array}$$

stalk

germ

representative for the germ

If we consider $\mathcal{F}(U)$ is a set of maps $\{U \to \text{target space}\}$ then we have:

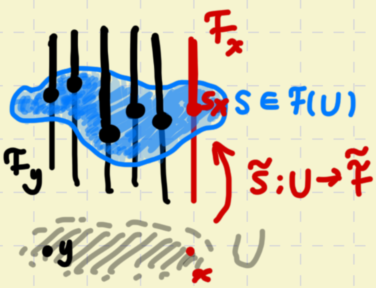


If $s(x) = s'(x)$ then $s_x = s'_x$.

[Construction] Let $\tilde{\mathcal{F}} = \bigcup_{x \in X} \mathcal{F}_x$, and let $\pi : \tilde{\mathcal{F}} \to X$ by sending points in $\mathcal{F}_x$ to x. To make $\tilde{\mathcal{F}}$ an étalé space, all remains is to give $\tilde{\mathcal{F}}$ a topology and check $\pi : \tilde{\mathcal{F}} \to X$ is a local homeo.

For $x \in X$, consider open n.bh. U of x.

$\tilde{\mathcal{F}}$ locally



Stalks parametrized by points in $U \overset{\text{open}}{\subseteq} X$

$\tilde{s}(U) = \{s_x \mid x \in U\}$

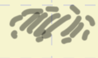
key: Endow topo of $\tilde{\mathcal{F}}$ by topo of X.

Fortunately, we can find a section so move U to $\tilde{\mathcal{F}}$ and let the image in $\tilde{\mathcal{F}}$ be open.

The section is easily find when we draw the left picture. For $s \in \mathcal{F}(U)$

let $\tilde{s} : U \to \tilde{\mathcal{F}}$, $x \mapsto s_x$.

Since $\pi \circ \tilde{s}(x) = \pi(s_x) = x$, so $\pi \circ \tilde{s} = id$ meaning that $\tilde{s}$ is a section, i.e., π is local bijection

In picture, it means  bijective to  U

Let $\{\tilde{s}(U) \mid U \overset{\text{open}}{\subseteq} X,\ s \in \mathcal{F}(U)\}$ be a basis for the topo of $\tilde{\mathcal{F}}$.
Then $\pi|_{\mathrm{Im}\,\tilde{s}}$ and its inverse $\tilde{s}$ are both conti, making π a local homeo.

[Exp] If the presheaf has algebraic properties preserved by direct limits, then the étalé space $\tilde{\mathcal{F}}$ inherits these props. For instance, suppose $\mathcal{F}$ is a presheaf of abelian grps.

① Each stalk $\mathcal{F}_x$ is an ab grp.

② Let $\tilde{\mathcal{F}} \circ \tilde{\mathcal{F}} = \{(s,t) \in \tilde{\mathcal{F}} \times \tilde{\mathcal{F}} \mid \pi(s) = \pi(t)\}$ (i.e., s,t lie in same stalk $\mathcal{F}_x$)

Define $\mu: \tilde{\mathcal{F}} \circ \tilde{\mathcal{F}} \longrightarrow \tilde{\mathcal{F}}$, $(s_x, t_x) \mapsto s_x - t_x$. It's well-defined since $s_x, t_x \in \mathcal{F}_x$ which is an ab grp. μ is a conti map, indeed, for $h \in \mathcal{F}(U)$, $\tilde{h}(U)$ is an open set in $\tilde{\mathcal{F}}$. Since $h \in \mathcal{F}(U)$ which is an ab grp, $\exists s,t$ in $\mathcal{F}(U)$ s.t. $h = s - t$. $\tilde{h}(U) = \widetilde{s-t}(U) = \{(s-t)_x \mid x \in U\} = \{s_x - t_x \mid x \in U\}$
So the inverse $\mu^{-1}(\tilde{h}(U)) = \{(s_x, t_x) \mid x \in U\} \subseteq \tilde{\mathcal{F}} \circ \tilde{\mathcal{F}}$, i.e.,

$$\tilde{s}(U) \circ \tilde{t}(U) = \{(a,b) \in \tilde{s}(U) \times \tilde{t}(U) \mid \pi(a) = \pi(b)\}$$
$$= \{(s_x, t_x) \mid x \in U\} = \mu^{-1}(\tilde{h}(U)).$$

So $\mu^{-1}(\tilde{h}(U)) = \tilde{s}(U) \circ \tilde{t}(U)$ is open in $\tilde{\mathcal{F}} \circ \tilde{\mathcal{F}}$.

③ $\Gamma(U, \tilde{\mathcal{F}})$ is an ab grp under pointwise addition, i.e., for $\tilde{s}, \tilde{t} \in \Gamma(U, \tilde{\mathcal{F}})$, $(\tilde{s} - \tilde{t})(x) = \tilde{s}(x) - \tilde{t}(x)$, $\forall x \in U$. Since $\tilde{s} - \tilde{t}$ is given by compositions:

$$U \xrightarrow{(\tilde{s},\tilde{t})} \tilde{\mathcal{F}} \circ \tilde{\mathcal{F}} \xrightarrow{\mu} \tilde{\mathcal{F}}$$
$$x \longmapsto (s_x, t_x) \longmapsto s_x - t_x$$

so $\tilde{s} - \tilde{t}$ is conti. □

Then we want to do the invers — given an étalé space, we want to associate it a sheaf. The natural choice is $\Gamma(-, \tilde{\mathcal{F}})$, the sheaf of sections of $\tilde{\mathcal{F}}$.

[Def] Let $\mathcal{F}$ be a presheaf over a topo space X and let $\bar{\mathcal{F}}$ be the sheaf of sections of the étalé space $\tilde{\mathcal{F}}$ associated with $\mathcal{F}$. Then we call $\bar{\mathcal{F}}$ is the sheaf generated by $\mathcal{F}$.

[Rmk] Sheafication is take sheaf of sections of étalé space. Étalé space is a good way pass from presheaf to sheaf.

Question: What's relationship between $\mathcal{F}$ and $\overline{\mathcal{F}}$? Let's find mors between them first. There is a presheaf mor $\tau: \mathcal{F} \to \overline{\mathcal{F}}$, with $\tau_U: \mathcal{F}(U) \to \overline{\mathcal{F}}(U) = \Gamma(U, \tilde{\mathcal{F}})$, $\tau_U(s) = \tilde{s}$. When $\mathcal{F}$ be a sheaf, we have:

[Thm] If $\mathcal{F}$ is a sheaf, then $\tau: \mathcal{F} \to \overline{\mathcal{F}}$ is a sheaf iso.

pf: It suffices to show $\tau_U: \mathcal{F}(U) \to \overline{\mathcal{F}}(U) = \Gamma(U, \tilde{\mathcal{F}})$ is bijective.

show τ_U is inj.: Suppose $a, b \in \mathcal{F}(U)$ s.t. $\tau_U(a) = \tau_U(b) \in \Gamma(U, \tilde{\mathcal{F}})$.
$\tau_U(a) = \tilde{a}: U \to \tilde{\mathcal{F}}$ with $\tilde{a}(x) = a_x = r^U_x a$ where $r^U_x: \mathcal{F}(U) \to \mathcal{F}_x$ is the data of $\varinjlim_{x \in U}$. Hence $\tau_U(a) = \tau_U(b)$ means $r^U_x a = r^U_x b$ for all $x \in U$.

Fact: For direct limit $\begin{array}{c} A_i \xrightarrow{f_{ij}} A_j \\ f_i \searrow \ \swarrow f_j \\ L \end{array}$, given any $x_1, x_2 \in A_i$ with $f_i(x_1) = f_i(x_2)$, there exists j s.t. $f_{ij}(x_1) = f_{ij}(x_2)$.

Hence, there exists open set $V_x \ni x$, s.t. $r^U_{V_x} a = r^U_{V_x} b$.

$$\begin{array}{c} a, b \\ \mathcal{F}(U) \xrightarrow{r^U_V} \mathcal{F}(V) \\ r^U_x \searrow \quad \swarrow r^U_x \\ \mathcal{F}_x \end{array}$$

$U = \bigcup_{x \in U} V_x$, $r^U_{V_x} a = r^U_{V_x} b$ means $a = b \in \mathcal{F}(U)$ by axiom S_1 of sheaf.

Show τ_U is surj.: $\tau_U: \mathcal{F}(U) \to \overline{\mathcal{F}}(U) = \Gamma(U, \tilde{\mathcal{F}})$.
Let $\sigma \in \Gamma(U, \tilde{\mathcal{F}})$. Pick $x \in U$, we have $\sigma(x) \in \mathcal{F}_x$. By direct limit property, there exist a n.b.h. $V \ni x$ and $s \in \mathcal{F}(V)$, s.t. $r^V_x s = \sigma(x)$. Since $r^V_x s = s_x = \tilde{s}(x) = \tau_V(s)(x)$, we have $\tau_V(s)(x) = \sigma(x)$. σ and $\tau_V(s)$ are sections of étalé space, and sections have local inverse π, hence any two sections of étalé space agree at one point will agree at a n.b.h. So there exists a n.b.h. W of x, s.t. $\sigma|_W = \tau_V(s)|_W = \tau_W(r^V_W s)$, the last equation is because

direct limit
$\begin{array}{c} A_i \to A_j \\ f_i \searrow \ \swarrow f_j \\ L \end{array}$
$\forall b \in L$ $\exists i$ and $a \in A_i$ s.t. $f_i a_i = b$

τ is a sheaf mapping:
$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\tau_V} & \overline{\mathcal{F}}(V) \\ r^V_W \downarrow & \circlearrowright & \downarrow r^V_W \\ \mathcal{F}(W) & \xrightarrow{\tau_W} & \overline{\mathcal{F}}(W) \end{array}$$

The above process can be done for any $x \in U$, hence we can find an open cover $\{U_i\}$ of U and $s_i \in \mathcal{F}(U_i)$ s.t. $\sigma|_{U_i} = \tau_{U_i}(s_i)$

(Replacing w to U_i and $r^V_W s$ to s_i .)

We want to find $s \in \mathcal{F}(U)$ s.t. $\tau_U(s) = \sigma$, i.e. $\tau_U(s)|_{U_i} = \sigma|_{U_i} = \tau_{U_i}(s_i)$.

So it suffices to find $s \in \mathcal{F}(U)$ s.t. $\tau_U(s)|_{U_i} = \tau_{U_i}(s_i)$. Play same trick of commutative diagram:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\tau_U"] \arrow[d, "r^U_{U_i}"'] \arrow[dr, phantom, "\circlearrowright"] & \bar{\mathcal{F}}(U) \arrow[d, "r^U_{\bar{\mathcal{F}}\,U_i}"] \\ \mathcal{F}(U_i) \arrow[r, "\tau_{U_i}"'] & \bar{\mathcal{F}}(U_i) \end{tikzcd}$$

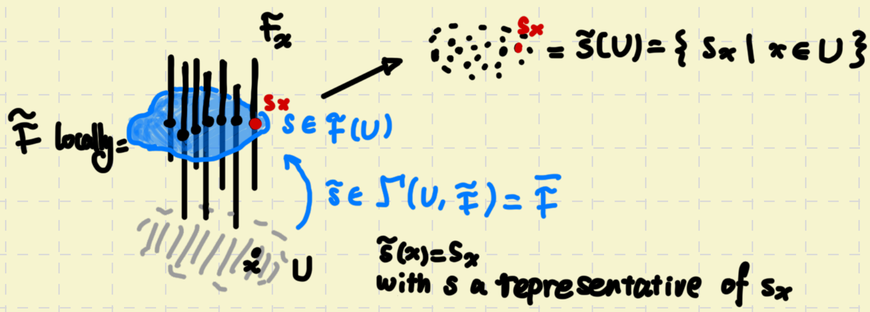
, we obtain $\tau_U(s)|_{U_i} = \tau_{U_i}(r^U_{U_i} s)$ for any $s \in \mathcal{F}(U)$

So we suffices to find $s \in \mathcal{F}(U)$ s.t. $r^U_{U_i} s = s_i$. It's easy to find s by glueing. $\tau_{U_i \cap U_j}(r^{U_i}_{U_i \cap U_j} s_i) = \sigma|_{U_i \cap U_j} = \tau_{U_i \cap U_j}(r^{U_j}_{U_i \cap U_j} s_j)$ and $\tau_{U_i \cap U_j}$ is injective, we have $r^{U_i}_{U_i \cap U_j} s_i = r^{U_j}_{U_i \cap U_j} s_j$. Since $\mathcal{F}$ is a sheaf and $U = \bigcup_i U_i$, there exists $s \in \mathcal{F}(U)$ s.t. $r^U_{U_i}(s) = s_i$. By above analysis, we complete the proof. $\square$

[Rmk] For a sheaf $\mathcal{F}$, find étalé space $\tilde{\mathcal{F}}$ and then take $\bar{\mathcal{F}} = \Gamma(-, \tilde{\mathcal{F}})$. The thm tells you $\mathcal{F} \cong \bar{\mathcal{F}}$, so $\bar{\mathcal{F}}$ contains inf. (information) of $\mathcal{F}$. $\tilde{\mathcal{F}}$ contains inf. of $\bar{\mathcal{F}}$, so $\tilde{\mathcal{F}}$ contains inf. of $\mathcal{F}$. But $\tilde{\mathcal{F}}$ is constructed from $\mathcal{F}$, so $\mathcal{F}$ also contains inf. of $\tilde{\mathcal{F}}$. In conclusion, the étalé space contains same amount inf. as sheaf $\mathcal{F}$ — hence, a sheaf is very often defined to be an étalé space with algebraic structure along its fibers. But when we encounter presheaf, the associated étalé space is an auxiliary construction.

[Rmk] For sheaf $\mathcal{F}$, we may not distinguish $\mathcal{F}$ and $\tilde{\mathcal{F}}$, i.e., we may identify two notations $\mathcal{F}(U)$ and $\Gamma(U, \tilde{\mathcal{F}})$ in some cases.

[Rmk] Relationship between $\mathcal{F}, \tilde{\mathcal{F}}, \bar{\mathcal{F}}$.



[Slogan] stalks remain unchanged by sheafication

$$\tilde{\mathcal{F}}_x = \varinjlim_{x\in U} \Gamma(U, \tilde{\mathcal{F}}) = \varinjlim_{x\in U} \Gamma(U, \bigcup_{y\in X} \mathcal{F}_y) = \mathcal{F}_x$$

[Construction] We've known $\mathcal{F}_x = \varinjlim_{x\in U} \mathcal{F}(U)$. Actually there is a concrete construction for $\mathcal{F}_x$, that is: $\mathcal{F}_x = \coprod_{U\ni x} \mathcal{F}(U)/\sim$ where $(f,V)\sim(g,W)$ iff there is an open $x\in H \subseteq V\cap W$ s.t. $r^V_H f = r^W_H g$.

1. Given a sheaf mor $\varphi: \mathcal{F} \to \mathcal{G}$, it induces a stalk mapping $\varphi_x: \mathcal{F}_x \to \mathcal{G}_x$ by $\varphi_x[(f,U)] = [\varphi_U(f), U]$ where $[\cdot]$ means equivalence class.
2. Let $\varphi: \mathcal{F} \to \mathcal{G}$, $\psi: \mathcal{F} \to \mathcal{G}$ be sheaf mors. Then $\varphi = \psi$ iff $\varphi_x = \psi_x$ for all $x \in X$.
3. $\ker(\varphi_x) = (\ker\varphi)_x$.

More details: https://web.ma.utexas.edu/users/slaoui/notes/Sheaf_Cohomology_3.pdf

The rest part is about exactness in homological algebra.

[Def] Let $\mathcal{F}, \mathcal{G}$ be sheaves of abelian grps over space X with $\mathcal{G}$ a subsheaf of $\mathcal{F}$. Let $\mathcal{Q}$ be the sheaf generated by the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{G}(U)$. Then $\mathcal{Q}$ is called the quotient sheaf of $\mathcal{F}$ by $\mathcal{G}$ and denoted by $\mathcal{F}/\mathcal{G}$.

[Rmk] $\mathcal{Q}$ is the sheafication of the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{G}(U)$, hence, $\mathcal{Q}(U) = \mathcal{F}/\mathcal{G}(U) \neq \mathcal{F}(U)/\mathcal{G}(U)$.

[Construction] Let's construct a natural sheaf surjection $\mathcal{F} \to \mathcal{F}/\mathcal{G}$. One may think it's surj projections $\mathcal{F}(U) \to \mathcal{F}(U)/\mathcal{G}(U)$, but note that $\mathcal{F}/\mathcal{G}(U) \neq \mathcal{F}(U)/\mathcal{G}(U)$, so there still remains some work. Denot $\mathcal{H}$ be the presheaf $[U \mapsto \mathcal{F}(U)/\mathcal{G}(U)]_U$. Consider the presheaf map $\tau: \mathcal{F} \to \mathcal{H}$ with $\tau_U: \mathcal{F}(U) \to \mathcal{F}(U)/\mathcal{G}(U)$. It induces a map between stalks $\tau_x: \mathcal{F}_x \to \mathcal{H}_x$ by going to direct limit:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] \arrow[rd] & \mathcal{F}(V) \arrow[ld] \arrow[rd] & \\
\mathcal{F}_x \arrow[rd, dashed, "\exists!", "\tau_x"'] & \mathcal{H}(U) \arrow[d] & \mathcal{H}(V) \arrow[ld] \\
& \mathcal{H}_x &
\end{tikzcd}$$

Then we induce a conti mapping of étalé spaces: $\tilde{\tau}: \tilde{\mathcal{F}} \to \tilde{\mathcal{H}}$, $x \mapsto \tau_x(x)$.

Consider the map induced on sections:

$$\tilde{\tau}_U: \Gamma(U, \tilde{\mathcal{F}}) \to \Gamma(U, \tilde{\mathcal{H}}), \quad s \mapsto \tilde{\tau}\circ s$$

It's well defined, just consider: A, B be étalé spac

$$\begin{tikzcd}
U \arrow[r, bend left, "s"] & A \arrow[l, "\pi_1"'] \arrow[d, "h"] \\
& B \arrow[lu, "\pi_2"]
\end{tikzcd}$$

$\pi_2 h s = \pi_1 s = id$, for $\forall s \in \Gamma(U, A)$ so $hs \in \Gamma(U, B)$.

This is the desired sheaf mapping onto the quotient sheaf. $\square$

[Def] (Exactness) If $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are sheaves of abelian grps over X and

$$\mathcal{A} \xrightarrow{g} \mathcal{B} \xrightarrow{h} \mathcal{C}$$ is a sequence of sheaf mors, then this sequence is exact at $\mathcal{B}$ if the induced sequence on stalks

$$\mathcal{A}_x \xrightarrow{g_x} \mathcal{B}_x \xrightarrow{h_x} \mathcal{C}_x$$ is exact for all $x \in X$. A short exact sequence is a sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ which is exact at $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$, where 0 denotes the constant zero sheaf.

[Rmk] Abelian property can pass to direct sum. So stalks are also abelian grps.

[Rmk] One may ask, why don't we define exact at $\mathcal{B}$ by exactness of the sequence $\mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U)$ for each open U? That's because exactness is a local property. Locally exact $\mathcal{A}_x \to \mathcal{B}_x \to \mathcal{C}_x$ doesn't mean globally exact $\mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U)$. The usefulness of sheaf theory is precisely in finding and categorizing obstructions to the "global exactness" of sheaves.

[Exp] X is a connected complex mfd.. Let $\mathcal{O}$ be the sheaf of holomorphic functions on X and let $\mathcal{O}^*$ be the sheaf of nonvanishing holomorphic functions on X which is a sheaf of ab grps under multiplication. (Nonvanishing implies we can do division, which makes $\mathcal{O}^*$ a sheaf of ab grps). Consider the sequence:

$$0 \to \mathbb{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

where $\mathbb{Z}$ is the constant sheaf $\mathbb{Z}(U) = \mathbb{Z}$, i is the inclusion map and $\exp: \mathcal{O} \to \mathcal{O}^*$ is $\exp_U : \mathcal{O}(U) \to \mathcal{O}^*(U)$, $f \mapsto \exp_U(f)$ with $\exp_U(f)(z) = \exp(2\pi i f(z)), \forall z \in U$ (nonvanishing on U)

To show this sequence is exact, we want to show at each $x \in X$,

$$0 \to \mathbb{Z}_x = \mathbb{Z} \xrightarrow{i_x} \mathcal{O}_x \xrightarrow{\exp_x} \mathcal{O}^*_x \to 0$$ is exact.

$\operatorname{Im} i_x = \mathbb{Z}$, so it remains to check $\ker(\exp_x) = \mathbb{Z}$.

Use conrete construct for stalks

$$\mathcal{O}_x \xrightarrow{\exp_x} \mathcal{O}^*_x$$
$$[(f, U)] \longmapsto [\exp_U(f), U]$$

($\mathcal{O}^*$ is a group with multiplication, so unit is germ 1_x)

Let $[\exp_U(f), U] = 1_x \in \mathcal{O}^*_x$, i.e. $[\exp(2\pi i f), U] = 1_x = [(1, U)]$. By def of equivalence class, there exists n.b.h. $V \subseteq U$ s.t. $\exp(2\pi i f(x)) = 1$, $\forall x \in V$. So $f(x)$ is a constant map on V, i.e.,

$[(f,U)] = [(l,V)]$, $l \in \mathbb{Z}$. Hence $\ker(\exp_x) = \mathbb{Z}$. $\square$.

[Exp] Let $\mathcal{A}$ be a subsheaf of $\mathcal{B}$. Then $0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \to \mathcal{B}/\mathcal{A} \to 0$ is an exact sequence of sheaves. (Note that only can sheaf of ab grp can do quotient, so $\mathcal{A}, \mathcal{B}$ are sheaves of ab grps, although we do not explicity state it).

pf: [Fact]: Colimit $\varinjlim$ in abelian category preserves exactness.

Since $0 \to F(U) \to G(U) \to G(U)/F(U) \to 0$ are exact sequence of ab grps, we have $0 \to \varinjlim_{x\in U} F(U) \to \varinjlim_{x\in U} G(U) \to \varinjlim_{x\in U} G(U)/F(U) \to 0$

i.e., $0 \to F_x \to G_x \to \mathcal{H}_x \to 0$ is exact, where $\mathcal{H}$ is presheaf $U \mapsto F(U)/G(U)$

Since stalks remain unchanged under sheafification, we have

$0 \to F_x \to G_x \to \underbrace{(F/G)_x}_{}^{\cong \mathcal{H}_x} \to 0$ is exact. Hence sheaf sequence

$0 \to F \to G \to F/G \to 0$ is exact.

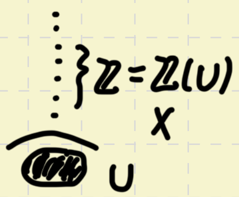
[Exp] Let $X = \mathbb{C}$ and $\mathcal{O}$ be the holomorphic functions on $\mathbb{C}$. Let $\mathcal{J}$ be the subsheaf of $\mathcal{O}$ consisting of holomorphic functions vanishing at $z=0 \in \mathbb{C}$. Then by the above example, $0 \to \mathcal{J} \to \mathcal{O} \to \mathcal{O}/\mathcal{J} \to 0$ is exact sequence of sheaves.

At $z \neq 0$, the sequence is $0 \to \mathbb{C} \to \mathbb{C} \to 0 \to 0$

At $z = 0$, the sequence is $0 \to 0 \to \mathbb{C} \to \mathbb{C} \to 0$

[Exp] X is a connected Hausdorff space and $a, b \in X$ fulfilling $a \neq b$. Let $\mathbb{Z}$ denote the constant sheaf of integers, i.e. $\mathbb{Z}(U) = \mathbb{Z}$. Let $\mathcal{J}$ denote the subsheaf of $\mathbb{Z}$ wich vanishes at a and b, that means $i_U: \mathcal{J}(U) \to \mathbb{Z}(U)$ is an inclusion with $i_U(a) = i_U(b) = 0$ for each U

sheaf $\mathbb{Z}$



Then we have exact seq

$0 \to \mathcal{J} \to \mathbb{Z} \to \mathbb{Z}/\mathcal{J} \to 0$

If $x=a$ or $x=b$, the seq of stalks is $0 \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0$

If $x \neq a$ and $x \neq b$, the seq of stalks is $0 \to \mathbb{Z} \to \mathbb{Z} \to 0 \to 0$ $\square$

The following sheaf means sheaf of ab grps or sheaf of modules.

[Def] A graded sheaf is a family of sheaves indexed by integers, $\mathcal{F}^* = \{\mathcal{F}^\alpha\}_{\alpha\in\mathbb{Z}}$. A sequence of sheaves (or sheaf sequence) is a graged sheaf connected by sheaf mappings:

$$\cdots \to \mathcal{F}^0 \xrightarrow{\alpha_0} \mathcal{F}^1 \xrightarrow{\alpha_1} \mathcal{F}^2 \xrightarrow{\alpha_2} \mathcal{F}^3 \to \cdots \quad (*)$$

A differential sheaf is a sequence of sheaves where $\alpha_j \alpha_{j-1} = 0$ in $(*)$. A resolution of a sheaf $\mathcal{F}$ is an exact sequence of sheaves of the form

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots \to \mathcal{F}^m \to \cdots$$

which we also denote symbolically by $0 \to \mathcal{F} \to \mathcal{F}^*$

[Rmk] Various type of information for a given sheaf $\mathcal{F}$ can be obtained from knowledge of a given resolution. Besides, resolution can be used in computing cohomology demonstrated next section.

[Exp] Let X be a differentiable m.f. of real dimension m and let $\mathcal{E}_X^p$ be the sheaf of real-valued differential form. We'll prove

$$0 \to \mathbb{R} \xrightarrow{i} \mathcal{E}_X^0 \xrightarrow{d} \mathcal{E}_X^1 \xrightarrow{d} \cdots \to \mathcal{E}_X^m \to 0$$

is a resolution of sheaf $\mathbb{R}$.

Fact: On a star-shaped domain U in $\mathbb{R}^n$, if $f \in \mathcal{E}^p(U)$ with $df = 0$, then there exists $u \in \mathcal{E}^{p-1}(U)$ $(p > 0)$ s.t. $du = f$.

For any $x \in X$, find a star-shaped domain U of x. Consider seq

$$0 \to \mathbb{R}(U) = \mathbb{R} \xrightarrow{i_U} \mathcal{E}_X^0(U) \xrightarrow{d} \mathcal{E}_X^1(U) \xrightarrow{d} \cdots \to \mathcal{E}_X^m(U) \to 0$$

It's exact at $\mathcal{E}_X^p(U)$, $p \geq 1$. By fact, $\ker d \subseteq \operatorname{Im} d$. By $d^2 = 0$, $\ker d \supseteq \operatorname{Im} d$. So $\ker d = \operatorname{Im} d$.

It's exact at $\mathcal{E}_0^p(U)$. $\mathbb{R} \xrightarrow{i} \mathcal{E}_X^0(U) = C^\infty(U, \mathbb{R}) \xrightarrow{d} \mathcal{E}_X^1(U) = \{f = \sum_i g_i dx_i \mid g_i \in C^\infty(U)\}$

$f \in \ker d \Leftrightarrow df = \sum_i \frac{\partial f}{\partial x_i} dx_i = 0 \Leftrightarrow \frac{\partial f}{\partial x_i} = 0$ on $U \Leftrightarrow f|_U \in \mathbb{R}$ is a const map

$\Leftrightarrow f \in \operatorname{Im} i$ Hence it's exact.

All in all, the sequence passing to stalks are also exact.

[Exp] X is a topo m.f. and G is an abelian grp. We want to derive a resolution for the constant sheaf of G over X.

Denote $S_p(U, \mathbb{Z})$ the abelian grp of integral singular chains of degree p in U, i.e., $S_p(U, \mathbb{Z}) = \{\sum a_i n_i \mid a_i \in \mathbb{Z}, n_i : \Delta^p \to U\}$. ($C_p(U)$ in Hatcher)

Denote $S^p(U, G) = \operatorname{Hom}_{\mathbb{Z}}(S_p(U, \mathbb{Z}), G)$ which is the group of singular cochains in U with coefficients in G. Let δ denote the coboundary operator, $\delta: S^p(U,G) \to S^{p+1}(U,G)$.

Let $S^p(G)$ be the sheaf over X generated by the presheaf $U \mapsto S^p(U,G)$ with induced differential mapping $S^p(G) \xrightarrow{\delta} S^{p+1}(G)$.

[How to induce this mapping? Rephrase our question is alwayse useful. $S^p(-,G)$, $S^{p+1}(-,G)$ are presheaves. We've know $\delta: S^p(-,G) \to S^{p+1}(-,G)$ given by coboundary mapping $\delta_U: S^p(U,G) \to S^{p+1}(U,G)$. We want to induce a sheaf map $\bar{\delta}: \bar{S}^p(-,G) \to \bar{S}^{p+1}(-,G)$. Here're detailed steps:

① Induce mapping between stalks $\delta_x: S^p_x(-,G) \to S^{p+1}_x(-,G)$

② Induce mapping between étalé space $\tilde{\delta}: \tilde{S}^p(-,G) \to \tilde{S}^{p+1}(-,G)$
$x \mapsto \delta_x(x)$

③ Induce mapping between sections $\bar{\delta}: \Gamma(-, \tilde{S}^p(-,G)) \to \Gamma(-, \tilde{S}^{p+1}(-,G))$]

Consider the unit ball U in Euclidean space. By alg topo, we've computed $H^*(U;G) = \begin{cases} G & * = 0 \\ 0 & * > 0 \end{cases}$. That means the seq

$$0 \to G \xrightarrow{i} S^0(U,G) \xrightarrow{\delta^0} \cdots \to S^{p-1}(U,G) \xrightarrow{\delta^{p-1}} S^p(U,G) \xrightarrow{\delta} S^{p+1}(U,G) \to \cdots$$

is exact. ($\ker \delta^0 = G$ by cohomology). Hence it's exact passing to any x in U. So the seq

$$0 \to G \to S^0(G) \xrightarrow{\delta} S^1(G) \xrightarrow{\delta} S^2(G) \to \cdots \to S^m(G) \to \cdots$$

is a resolution of const sheaf G, which we abbreviate by $0 \to G \to S^*(G)$.

We could also consider C^∞ chains and similary obtain a resolution $0 \to G \to S^*_\infty(G)$. ($0 \to G \to S^0_\infty(G) \to \cdots \to S^m_\infty(G) \to \cdots$)

[Exp] X is a complex m.f. of complex dimension n. Let $\mathcal{E}^{p,q}$ be the sheaf of (p,q) forms on X. Consider the sequence of sheaves in which $p \geq 0$ fixed:

$$0 \to \Omega^p \xrightarrow{i} \mathcal{E}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1} \xrightarrow{\bar{\partial}} \cdots \to \mathcal{E}^{p,n} \to 0$$

where Ω^p is defined as the kernel sheaf of the mapping $\mathcal{E}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1}$. kernel sheaf Ω^p is the subsheaf of $\mathcal{E}^{p,0}$, hence Ω^p is the sheaf of holomorphic differential forms of type $(p,0)$, i.e., $\varphi \in \Omega^p(U)$ has the form $\varphi = \sum'_{|I|=p} \varphi_I dz^I$, $\varphi_I \in \mathcal{O}(U)$. For each p, we have a resolution of Ω^p : $0 \to \Omega^p \to \mathcal{E}^{p,*}$. The proof use $\bar{\partial}^2 = 0$ and Grothendick version of the Poincaré Lamma for the $\bar{\partial}$-operator. Detailed proof is similar in proving resolution $0 \to \mathbb{R} \to \mathcal{E}^*$. Statement of the Grothendick version of the Poincaré lemma for the $\bar{\partial}$-operator: If ω is a (p,q)-form defined in a polydisc Δ in $\mathbb{C}^n$ where $\Delta = \{z \mid |z_i| < r,\ i = 1, \cdots, n\}$, and $\bar{\partial}\omega = 0$ in Δ, then there exists a $(p, q-1)$-form u defined in a slightly smaller polydisc $\Delta' \subset\subset \Delta$ so that $\bar{\partial} = \omega$ in Δ'.

[Exp] X is a complex m.f. . Ω^p is the kernel sheaf of sheaf mapping $\mathcal{E}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1}$. Consider sheaf sequence

$$0 \to \mathbb{C} \to \Omega^0 \xrightarrow{\partial} \Omega^1 \to \cdots \xrightarrow{\partial} \Omega^n \to 0$$

($\partial : \mathcal{E}^{p,0} \to \mathcal{E}^{p+1,0}$, $\Omega^p \subseteq \mathcal{E}^{p,0}$ so we have $\partial : \Omega^p \to \mathcal{E}^{p+1,0}$. Since $\partial\bar{\partial} + \bar{\partial}\partial = 0$, we have $\partial\overset{0}{\bar{\partial}\Omega^p} + \bar{\partial}\partial\Omega^p = 0$ so $\bar{\partial}\partial\Omega^p = 0$. Hence $\partial\Omega^p \subseteq \ker\bar{\partial}$. Therefore we have $\underset{\text{Since } \Omega^p = \ker\bar{\partial}}{\partial : \Omega^p \to \Omega^{p+1}}$)

We claim it's a resolution of $\mathbb{C}$ without proof. $\square$

[Def] Let $\mathcal{L}^*$ and $\mathcal{M}^*$ be differential sheaves. Then a homomorphism $f : \mathcal{L}^* \to \mathcal{M}^*$ is a sequence of holomorphism $f_j : \mathcal{L}^j \to \mathcal{M}^j$ which commutes with the differentials of $\mathcal{L}^*$ and $\mathcal{M}^*$.
A holomorphism of resolution of sheaves is a homomorphism of the underlying differential sheaves.

$$\begin{array}{ccccc} 0 \to & \mathcal{A} & \to & \mathcal{A}^* \\ & \downarrow & & \downarrow \\ 0 \to & \mathcal{B} & \to & \mathcal{B}^* \end{array}$$

[Exp] X is a differentiable m.f. and let

$$0 \to \mathbb{R} \to \mathcal{E}^*,\ 0 \to \mathbb{R} \to S^*_\infty(\mathbb{R})$$ be the resolutions

of $\mathbb{R}$ given by previous examples. Define $I : \mathcal{E}^* \to S^*_\infty(\mathbb{R})$ by setting $I_U : \mathcal{E}^*(U) \to S^*_\infty(U, \mathbb{R})$

$$\varphi \longmapsto I_U(\varphi) \text{ which is } I_U(\varphi)(c) = \int_c \varphi$$

It induces a map of resolutions

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r, "i"] \arrow[d, "id"'] & \mathcal{E}^* \arrow[d, "I"] \\ 0 \arrow[r] & \mathbb{R} \arrow[r, "i"] & S^*_\infty(\mathbb{R}) \end{tikzcd}$$

To show it's a homomorphism, we only need to show the diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r, "i"] \arrow[d, "id"] \arrow[dr, phantom, "①"] & \mathcal{E}^0 \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{E}^p \arrow[r] \arrow[d] \arrow[dr, phantom, "②"] & \mathcal{E}^{p+1} \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & \mathbb{R} \arrow[r, "i"] & S^0_\infty \arrow[r] & \cdots \arrow[r] & S^p_\infty(\mathbb{R}) \arrow[r] & S^{p+1}(\mathbb{R}) \arrow[r] & \cdots \end{tikzcd}$$

For ①:

$$\begin{tikzcd} r \in \mathbb{R} \arrow[r, "i"] \arrow[dr, "i"'] & \mathcal{E}^0(U) = C^\infty(U, \mathbb{R}) \arrow[d, "I_U"] \\ & S^0_\infty(U, \mathbb{R}) = \mathrm{Hom}(S_0(U;\mathbb{R}), \mathbb{R}) \end{tikzcd}$$

$r \mapsto \varphi = \left[\begin{matrix} U \to \mathbb{R} \\ x \mapsto r \end{matrix}\right] \mapsto I_U(\varphi) = [c \mapsto \underset{\overset{\shortparallel}{r}}{\int_c \varphi}]$

$r \mapsto [c \mapsto r]$

$=$

For ②

$$\begin{tikzcd} \varphi \in \mathcal{E}^p(U) \arrow[r] \arrow[d] & \mathcal{E}^{p+1}(U) \arrow[d] \\ S^p_\infty(U, \mathbb{R}) \arrow[r, "\delta"] & S^{p+1}_\infty(U, \mathbb{R}) \end{tikzcd}$$

$\varphi \mapsto d\varphi$

$\varphi \mapsto \psi = [c \mapsto \int_c \varphi] \longmapsto \delta\psi = [c \mapsto \int_c d\varphi]$

$$\begin{aligned} \delta\psi(c) &= \psi(\partial c) \\ &= \int_{\partial c} \varphi \\ &\overset{\text{stokes}}{=} \int_c d\varphi \end{aligned}$$

So $\delta\psi = [c \mapsto \int_c d\varphi]$

[Prop] Suppose $\varphi \in \mathcal{E}^{p,q}(U)$ for U open in $\mathbb{C}^n$ and $d\varphi = 0$. Then for any point $p \in U$, there is a n.b.h. N of p and a differential form $\psi \in \mathcal{E}^{p-1,q-1}(N)$ s.t. $\partial\bar{\partial}\psi = \varphi$ in N.

pf: key: application of Poincaré lemmas for the operators d, ∂, and $\bar{\partial}$.

$\mathcal{E}^{r-1}_x \xrightarrow{d} \mathcal{E}^r_x \xrightarrow{d} \mathcal{E}^{r+1}_x$ is exact, so $d\varphi = 0$ means there is $u \in \mathcal{E}^{r-1}_x$ s.t. $du = \varphi$, where $r = p+q$ is the total degree of φ.

Write $u = u^{r-1,0} + \cdots + u^{0,r-1}$, then $du = (\partial+\bar\partial)u = \underbrace{u^{r,0}}_{du^{r-1,0}} + u^{r-1,1} + \cdots$

But $du = \varphi$ which is a (p,q)-form, hence we only have these terms:

$du = \partial u^{p-1,q} + \bar\partial u^{p,q-1}$. Since $\bar\partial u^{p-1,q} = \partial u^{p,q-1} = 0$, we can apply $\bar\partial$ and ∂ Poincaré lemmas, so there are $\psi_1, \psi_2 \in \mathcal{E}_X^{p-1,q-1}$ s.t. $\partial\psi_1 = u^{p,q-1}$ and $\bar\partial\psi_2 = u^{p-1,q}$. Hence, we have

$$\begin{aligned}\varphi = du &= \partial u^{p-1,q} + \bar\partial u^{p,q-1} \\ &= \partial\bar\partial\psi_2 + \bar\partial\partial\psi_1 \\ &= \partial\bar\partial(\psi_2 - \psi_1)\end{aligned}$$

$\partial\bar\partial + \bar\partial\partial = 0$

Cohomology theory

In this section, we'll see how resolutions can be used to represent the cohomology groups of a space. In particular, we shall see every sheaf admits a canonical resolution with certain nice (cohomological) properties.

[Fact] For a short exact sequence of sheaves over X

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

Take its value at X, we have a sequence

$$0 \to \mathcal{A}(X) \to \mathcal{B}(X) \to \mathcal{C}(X) \to 0$$

This sequence is exact at $\mathcal{A}(X)$ and $\mathcal{B}(X)$ but not necessarily at $\mathcal{C}(X)$.

[Exp] X is a connected Hausdorff space, let $a, b \in X$ and $a \neq b$. $\mathbb{Z}$ is the constant sheaf of integers on X and $\mathcal{J}$ denote the subsheaf of $\mathbb{Z}$ vanishing at a and b. We have exact seq $0 \to \mathcal{J} \to \mathbb{Z} \to \mathbb{Z}/\mathcal{J} \to 0$. Consider sequence

$$0 \to \mathcal{J}(X) \to \underset{\underset{\Gamma(X,\mathbb{Z}) := \Gamma(X,\tilde{\mathbb{Z}})}{\parallel}}{\mathbb{Z}(X)} \to \underset{\underset{\Gamma(X,\widetilde{\mathbb{Z}/\mathcal{J}}) =: \Gamma(X,\mathbb{Z}/\mathcal{J})}{\cong}}{\mathbb{Z}/\mathcal{J}(X)} \to 0$$

$\forall f \in \Gamma(X,\mathbb{Z})$, $f(a) = f(b)$. $\forall g \in \Gamma(X, \mathbb{Z}/\mathcal{J})$, $g(a)$ may not equal to $g(b)$

So $\mathbb{Z}(X) \to \mathbb{Z}/\mathcal{J}(X)$ is not surj.

Cohomology gives a measure to the amount of inexactness of the sequence at $\mathcal{C}(X)$.

[Construction] Let $\mathcal{F}$ be a sheaf over a space X and let S be a closed subset of X. Define

$$\mathcal{F}(S) := \varinjlim_{U \supset S} \mathcal{F}(U) .$$

We've shown the sheaf mor $\tau: \mathcal{F} \to \bar{\mathcal{F}} = \Gamma(-, \tilde{\mathcal{F}})$ is an iso. Hence $\mathcal{F}(S)$ can be identified with $\Gamma(S, \tilde{\mathcal{F}}) = \Gamma(S, \pi^{-1}(S) =: \tilde{\mathcal{F}}|_S)$ where $\pi: \tilde{\mathcal{F}} \to X$ is the étalé map. For simplicity, we denote $\mathcal{F}(S)$ by $\Gamma(S, \mathcal{F})$. $\square$

Note that: ① for any $s \in \mathcal{F}(S)$, there exists open set $U \supseteq S$, and exists $f \in \mathcal{F}(U) = \Gamma(U, \tilde{\mathcal{F}}|_U)$ s.t. $f|_S = s$. (Property of direct limit)

$$\begin{tikzcd} \exists f \arrow[drr, bend right] & \mathcal{F}(U) \arrow[rr] \arrow[dr, "r^U_S"'] & & \mathcal{F}(V) \arrow[dl, "r^V_S"] \\ & & \mathcal{F}(S) \ni s & \end{tikzcd}$$

Prop: Given a direct limit $A_i \xrightarrow{f_{ij}} A_j$, $f_i \searrow L \swarrow f_j$, for any $l \in L$, $\exists$ i and $a \in A_i$ s.t. $f_i a = l$. It's proved by pick image.

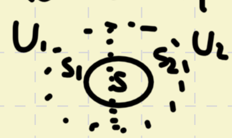
② For any $s \in \mathcal{F}(S)$, there exists an open covering $\{U_i\}$ of S and $s_i \in \mathcal{F}(U_i)$, s.t. $s_i|_{S \cap U_i} = s|_{S \cap U_i}$.

Indeed, we pick open $U \supseteq S$ s.t. there exists $f \in \mathcal{F}(U)$ with $f|_S = s|_S$. We decompose U to a union of open sets $\{U_i\}$. Let $f|_{U_i}$ denoted by s_i.

So we have $s_i|_{S \cap U_i} = f|_{U_i \cap S} = s|_{U_i \cap S}$ $\square$

① says that we can extend $s \in \mathcal{F}(S)$ to a section over an open set U

② says that we can decompose $s \in \mathcal{F}(S)$ under an open covering



$s_i|_{U_i \cap S} = s|_{U_i \cap S}$

From now on, we're dealing with sheaves of ab grp over a paracompact Hausdorff space X for simplicity.

[Def] A sheaf $\mathcal{F}$ over a space X is soft if for any closed $S \subseteq X$ the restriction mapping $\mathcal{F}(X) \to \mathcal{F}(S)$ is surj, i.e., any section of $\mathcal{F}$ over S can be extended to a section of $\mathcal{F}$ over X.

[Rmk] It's a kind of lifting property.

[Thm] If $\mathcal{A}$ is a soft sheaf and

$$0 \to \mathcal{A} \xrightarrow{g} \mathcal{B} \xrightarrow{h} \mathcal{C} \to 0$$

is a short exact seq of sheaves, then the induced seq

$$0 \to \mathcal{A}(X) \xrightarrow{g_x} \mathcal{B}(X) \xrightarrow{h_x} \mathcal{C}(X) \to 0$$

is exact.

pf: We only need to show it's exact at $\mathcal{C}(X)$.

$\Leftarrow$ Given $c \in \mathcal{C}(X)$, we need to find it's preimage under h_x in $\mathcal{B}(X)$.

- Find $\{b_i\}$s on $\{U_i\}$ in $\mathcal{B}(X)$. Since sheaf seq is exact, so for any $x \in X$, we have $h_x : \mathcal{B}_x \to \mathcal{C}_x$ is surj.

Hence, $\exists\, l \in \mathcal{B}_x$ s.t. $h_x l = r^X_x c \in \mathcal{C}_x$. By prop of direct limit, $\exists\, U$ open and $b \in \mathcal{B}(U)$ s.t. $r^U_x b = l \in \mathcal{B}_x$.

Consider the commutative diagram:

$$\begin{tikzcd}
b \arrow[d, mapsto] & \mathcal{B}(U) \arrow[r, "h_u"] \arrow[d, "r^U_x"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{C}(U) \arrow[d, "r^U_x"] & c|_U \arrow[ddl, mapsto] \\
l \arrow[drr, mapsto] & \mathcal{B}_x \arrow[r, "h_x"] & \mathcal{C}_x & \\
& & r^X_x c &
\end{tikzcd}$$

So $h_u b = c|_u$.

Therefore we can find an open cover of X $\{U_i\}$ and $b_i \in \mathcal{B}(U_i)$ s.t. $h_{u_i} b_i = c|_u$.

- Show $\{b_i\}$ can be pieced to a global section.

Since X is paracompact, $\exists$ locally finite refinement $\{S_i\}$ of $\{U_i\}$ s.t. S_i are closed set, $\forall i$. Consider the following set

$$P = \{(b, S) \mid S = \bigcup_{i \in I} S_i,\ b \in \mathcal{B}(S),\ h_s(b) = c|_s\}$$

P is partially ordered by $(b,S) \le (b', S')$ if $S \subseteq S'$ and $b'|_s = b$.

By Axiom S_2 of the sheaf, every linearly ordered chain has a maximal element by glueing. Hence by Zorn's lemma, there exists a maximal set S and a section $b \in \mathcal{B}(S)$ s.t. $h(b) = c|_S$. It remains to show $S = X$. Suppose on the contrary that there exists $S_j \in \{S_i\}$ s.t. $S_j \not\subseteq S$. If $S_j \cap S = \emptyset$, then we have $b' \in \mathcal{B}(S \cup S_j)$ by setting $b' = \begin{cases} b & x \in S \\ b_j & x \in S_j \end{cases}$, clearly $h(b')|_{S \cup S_j} = c|_{S \cup S_j}$ since $h(b)|_S = c|_S$ and $h(b_j)|_{S_j} = c|_{S_j}$. So S is not max, hence $S_j \cap S \neq \Phi$. Since $h(b)|_{S\cap S_j} = c|_{S\cap S_j} = h(b_j)|_{S\cap S_j}$, we have $h(b-b_j) = h(b) - h(b_j) = 0$ on $S_j \cap S$. By exactness at $\mathcal{A}(S\cap S_j) \xrightarrow{g} \mathcal{B}(S\cap S_j) \xrightarrow{h} \mathcal{C}(S\cap S_j)$, there exists $a \in \mathcal{A}(S\cap S_j)$ s.t. $g(a) = b - b_j$. Since $\mathcal{A}$ is soft, we extend a to a global section $\tilde{a}$. Define $\tilde{b} \in \mathcal{B}(S \cup S_j)$ by

$$\tilde{b} = \begin{cases} b & \text{on } S \\ b_j + g(\tilde{a}) & \text{on } S_j \end{cases} . \qquad (\text{on } S_j \cap S,\ b_j + g(\tilde{a}) = b_j + b - b_j = b)$$

Since $h(\tilde{b}) = c|_{S\cup S_j}$, S is not max. We complete the proof. □

[Def] A sheaf of abelian grps $\mathcal{F}$ over a paracompact Hausdorff space X is fine if for any locally finite open cover $\{U_i\}$ of X, there exists a family of sheaf mors

$$\{\eta_i : \mathcal{F} \to \mathcal{F}\}$$

s.t. (a) $\sum \eta_i = 1$

(b) $\eta_i(\mathcal{F}_x) = 0$ for all x in some n.b.h. of the complement of U_i

The family $\{\eta_i\}$ is called a partition of unity of subordinate to the covering $\{U_i\}$. □

[Rmk] 

$\forall x \in W \quad \eta_i(\mathcal{F}_x) = 0$. | We require W be n.b.h. of U_i^c, s.t. it's identically zero on U_i^c and a n.b.h. of ∂U_i. □

[Exp] Since partition of unity subordinate to any open cover is exist, so we have following fine sheaves:

1. $\mathcal{C}_X$ for X a paracompact Hausdorff space is a fine sheaf.
2. $\mathcal{E}_X$ for X a paracompact differentiable mf.
3. $\mathcal{E}_X^{p,q}$ for X a paracompact almost-complex mf.
4. A locally free sheaf of $\mathcal{E}_X$-modules, where X is a differentiable mf. ($5 \Rightarrow 4$)
5. If $\mathcal{R}$ is a fine sheaf of rings with unit, then any module over $\mathcal{R}$ is a fine sheaf. □

[prop] Fine sheaves are soft

pf: Let $\mathcal{F}$ be a fine sheaf over X and $S \overset{\text{closed}}{\subseteq} X$, $s \in \mathcal{F}(S)$. By def of soft, we w.t.s. the section s can be extended to a section over X. We hope to construct a section over X by glueing sections on open covering of X.

There is an open covering $\{U_i\}$ of S and sections $s_i \in \mathcal{F}(U_i)$ s.t. $s_i|_{S\cap U_i} = s|_{S\cap U_i}$. Let $U_0 = X - S$ and $s_0 = 0$, so that $\{U_i\} \cup U_0$ is an open covering of X. Since X is paracompact, we can assume $\{U_i\}$ is locally finite. Hence, by $\mathcal{F}$ soft, we have a partition of unity $\{\eta_i : \mathcal{F} \to \mathcal{F}\}$ subordinate to $\{U_i\}$. Consider $(\eta_i)_{U_i} : \mathcal{F}(U_i) \longrightarrow \mathcal{F}(U_i)$, we have $(\eta_i)_{U_i}(s_i) \in \mathcal{F}(U_i)$.
Since $(\eta_i)_{U_i}(s_i)|_{\text{n.b.h. } W \text{ of } U_i^c} = 0$, so $(\eta_i)_{U_i}(s_i)$ can be extend to a section over X, i.e., $(\eta_i)_{U_i}(s_i) \in \mathcal{F}(X)$.

Define $\tilde{s} = \sum_i (\eta_i)_{U_i}(s_i) \in \mathcal{F}(X)$, we'll show it's a section extended by $s \in \mathcal{F}(S)$, i.e., check $\tilde{s}|_S = s$

For $a \in S$, $\tilde{s}(a) = \sum_i (\eta_i)_{U_i}(s_i)(a) \underset{\uparrow\, (\eta_i)_{U_i}(s_i)|_{U_i^c} = 0}{=} \sum_{a \in U_i} (\eta_i)_{U_i}(s_i)(a) \overset{s_i(a) = s(a)}{=} \sum (\eta_i)_a (s)(a) \underset{\sum(\eta_i)_a = 1}{=} s(a)$.

[Exp] X be the complex and let $\mathcal{O} = \mathcal{O}_X$ be the sheaf of holomorphic functions on X. Let $S = \{|z| \le 1/2\}$. Let $f(z) = \sum z^{n!}$ on S. It cannot be extended to X. So $\mathcal{O}$ is not soft and hence not fine.

[Exp] Constant sheaf is not soft and hence not fine. Let G be constant sheaf over X and let $a, b \in X$ with $a \neq b$.
Define $s \in G(\{a,b\})$ by setting $s(a) = 0$ and $s(b) \neq 0$.
There doesn't exist $f \in G(X) = G$ s.t. $f|_{\{a,b\}} = s$, i.e., $f|_a = 0 \neq f|_b$ wich is impossible, because f is a fix element in G. Hence G is not soft and thus not fine.

[prop] For exact seq $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is exact with $\mathcal{A}, \mathcal{B}$ soft, then $\mathcal{C}$ is soft.

pf: Fix a closed set $S \subseteq X$. Since $\mathcal{A}$ is soft, we have the seq

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{A}(S) & \to & \mathcal{B}(S) & \xrightarrow{f} & \mathcal{C}(S) & \to & 0 \\ & & \uparrow r^X_{\mathcal{A}S} & & \uparrow r^X_{\mathcal{B}S} & & \uparrow r^X_{\mathcal{C}S} & & \\ 0 & \to & \mathcal{A}(X) & \to & \mathcal{B}(X) & \xrightarrow{g} & \mathcal{C}(X) & \to & 0 \end{array}$$

exact at $\mathcal{C}(S)$ and $\mathcal{C}(X)$.

For any $s \in \mathcal{C}(S)$, $\exists\, w \in \mathcal{B}(S)$ s.t. $f(w) = s$. Since $\mathcal{B}$ is soft, there exists $t \in \mathcal{B}(X)$ with $r^X_{\mathcal{B}\,\mathcal{S}}(t) = w$. Consider $g(t)$, by commutativity, $r^X_{\mathcal{C}\,\mathcal{S}}\, g(t) = s$. So we find suitable $r^X_{\mathcal{C}\,\mathcal{S}} \in \mathcal{C}(X)$ as an extension of s.

[prop] If $0 \to \mathcal{S}_0 \xrightarrow{f_0} \mathcal{S}_1 \xrightarrow{f_1} \mathcal{S}_2 \xrightarrow{f_2} \cdots$ is an exact sequence of soft sheaves, then the induced section sequence

$$0 \to \mathcal{S}_0(X) \to \mathcal{S}_1(X) \to \cdots$$

is also exact.

pf: Let $\mathcal{K}_i = \ker(\mathcal{S}_i \to \mathcal{S}_{i+1})$. We have short exact sequences

$$0 \to \mathcal{K}_i \xrightarrow{\iota} \mathcal{S}_i \xrightarrow{f_i} \mathcal{K}_{i+1} \to 0$$

($\operatorname{Im} f_i = \ker f_{i+1} = \mathcal{K}_{i+1}$. so f_i surj.)

key: Induction.

$i=1$ $\quad 0 \to \mathcal{K}_1 = f_0\mathcal{S}_0 = \mathcal{S}_0 \to \mathcal{S}_1 \xrightarrow{f_1} \mathcal{K}_2 \to 0$ exact.

With $\mathcal{S}_0, \mathcal{S}_1$ soft, we have $\mathcal{K}_2$ soft.

Suppose $\mathcal{K}_i$ is soft. For exact seq $0 \to \mathcal{K}_i \to \mathcal{S}_i \to \mathcal{K}_{i+1} \to 0$ with $\mathcal{K}_i, \mathcal{S}_i$ soft, we have $\mathcal{K}_{i+1}$ soft. Hence $\mathcal{K}_m$ soft for all m.

Since $\mathcal{K}_i$ is soft, we have short exact seqs

$$0 \to \mathcal{K}_i(X) \xrightarrow{\iota} \mathcal{S}_i(X) \xrightarrow{f_i} \mathcal{K}_{i+1}(X) \to 0.$$

Then we have a long exact seq by splicing thoes short exact seq.

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[dr] & \mathcal{S}_0(X) \arrow[rr, "\iota f_0"] \arrow[dr, "f_0"] & & \mathcal{S}_1(X) \arrow[rr, "\iota f_1"] \arrow[dr, "f_1"] & & \mathcal{S}_2(X) \arrow[dr, "f_2"] & \cdots \\
& \mathcal{K}_0(X) \arrow[u, "\iota"] & \mathcal{K}_1(X) \arrow[ur, "\iota"] & & \mathcal{K}_2(X) \arrow[ur, "\iota"] & & \mathcal{K}_3(X) \arrow[u, "\iota"]
\end{tikzcd}$$

[Construction] (Canonical soft resolution for any sheaf) Let $\mathcal{S}$ be a sheaf over X and let $\tilde{\mathcal{S}} \xrightarrow{\pi} X$ be the étalé space associated to $\mathcal{S}$.

Define a presheaf $C^0(\mathcal{S})(U) = \{ f : U \to \tilde{\mathcal{S}} \mid \pi \circ f = 1_U \}$. It's a sheaf and called the sheaf of discontinous sections of $\tilde{\mathcal{S}}$ over X.

Define sheaf mapping $h_0 : \mathcal{S} \to C^0(\mathcal{S})$ by $s \mapsto \tilde{s} \in \Gamma(U, \widetilde{C^0(\mathcal{S})})$ where $\tilde{s} : U \to \widetilde{C^0(\mathcal{S})}$, $x \mapsto s_x$. h_0 is injective, so we define $\mathcal{F}^1(\mathcal{S}) = C^0(\mathcal{S})/\mathcal{S}$ and $C^1(\mathcal{S}) = C^0(\mathcal{F}^1(\mathcal{S}))$. By induction, we define $\mathcal{F}^i(\mathcal{S}) = C^{i-1}(\mathcal{S})/\mathcal{F}^{i-1}(\mathcal{S})$ and $C^i(\mathcal{S}) = C^0(\mathcal{F}^i(\mathcal{S}))$. So we have

$$\left.\begin{aligned} &0 \to \mathcal{S} \to C^0(\mathcal{S}) \to \mathcal{F}^1(\mathcal{S}) \to 0 \\ &0 \to \mathcal{F}^i(\mathcal{S}) \to C^i(\mathcal{S}) \to \mathcal{F}^{i+1}(\mathcal{S}) \to 0 \end{aligned}\right\} \text{Both exact.}$$

Splicing them together, we obtain the long exact seq

$$0 \to \mathcal{S} \to \mathcal{C}^0(\mathcal{S}) \to \mathcal{C}^1(\mathcal{S}) \to \mathcal{C}^2(\mathcal{S}) \to \cdots$$

$\mathcal{F}^i(\mathcal{S})$ $\quad$ $\mathcal{F}^{i+1}(\mathcal{S})$

We call it the canonical resolution of $\mathcal{S}$ and abbreviate by

$$0 \to \mathcal{S} \to \mathcal{C}^*(\mathcal{S})$$

$\mathcal{C}^0(\mathcal{S})$ is soft if $\mathcal{S}$ is a sheaf, so $\mathcal{C}^i(\mathcal{S}) = \mathcal{C}^0(\mathcal{F}^i(\mathcal{S}))$ is soft since $\mathcal{F}^i(\mathcal{S})$ is a sheaf. Hence $0 \to \mathcal{S} \to \mathcal{C}^*(\mathcal{S})$ is a soft resolution.

Next, we need to define the cohomology grps of a space with <u>coefficients in a given sheaf</u>.

Let $\mathcal{S}$ be a sheaf over X and consider its canonical soft resolution

$0 \to \mathcal{S} \to \mathcal{C}^0(\mathcal{S}) \to \mathcal{C}^1(\mathcal{S}) \to \cdots$

Take global section X we have a seq by taking (continous) sections

$$0 \to \Gamma(X, \mathcal{S}) \to \Gamma(X, \mathcal{C}^0(\mathcal{S})) \to \Gamma(X, \mathcal{C}^1(\mathcal{S})) \to \cdots$$

[Rmk] One may feel confused about this notation.

$\Gamma(X, \mathcal{S}) := \Gamma(X, \tilde{\mathcal{S}})$, $\Gamma(X, \mathcal{C}^0(\mathcal{S})) := \Gamma(X, \widetilde{\mathcal{C}^0(\mathcal{S})})$.

Since $\mathcal{S}$ and $\mathcal{C}^i(\mathcal{S})$ are sheaves, we have $\Gamma(-, \mathcal{C}^i(\mathcal{S})) \cong \mathcal{C}^i(\mathcal{S})(-)$ and $\Gamma(-, \mathcal{S}) \cong \mathcal{S}(-)$.

[Rmk] If $\mathcal{S}$ is soft, then we have exact soft seq $0 \to \mathcal{S} \to \mathcal{C}^0(\mathcal{S}) \to \cdots$

Hence by previous property, we have exact seq

$$0 \to \underset{\mathcal{S}(X)}{\underset{\|}{\Gamma(X, \mathcal{S})}} \to \underset{\mathcal{C}^0(\mathcal{S})(X)}{\underset{\|}{\Gamma(X, \mathcal{C}^0(\mathcal{S}))}} \to \underset{\mathcal{C}^1(\mathcal{S})(X)}{\underset{\|}{\Gamma(X, \mathcal{C}^1(\mathcal{S}))}} \to \cdots \to \cdots$$

$\square$

[Def] Let $\mathcal{S}$ be a sheaf over a space X and let

$H^q(X, \mathcal{S}) := H^q(C^*(X, \mathcal{S}))$ where $H^q(C^*(X, \mathcal{S}))$ is the qth derived group of the cochain complex $C^*(X, \mathcal{S})$.

$(0 \to C^0(X, \mathcal{S}) \to C^1(X, \mathcal{S}) \to \cdots)$

The abelian groups $H^q(X, \mathcal{S})$ are defined for $q \geq 0$ and are called the sheaf cohomology groups of the space X of degree q and with coefficient in $\mathcal{S}$

[Rmk] This abstract definition is useful to derive general functorial properties of cohomology grps, and we have various other ways to do computations.

[Thm] Let X be a paracompact Hausdorff space. Then

(a) For any sheaf $\mathcal{S}$ over X,

(1) $H^0(X, \mathcal{S}) = \Gamma(X, \mathcal{S})\ (= \mathcal{S}(X))$

(2) If $\mathcal{S}$ is soft, then $H^q(X, \mathcal{S}) = 0$ for $q > 0$

(b) For any sheaf mor $h: \mathcal{A} \to \mathcal{B}$

there is, for each $q \geq 0$, a grp homo $h_q: H^q(X, \mathcal{A}) \to H^q(X, \mathcal{B})$

s.t. (1) $h_0 = h_X : \mathcal{A}(X) \to \mathcal{B}(X)$

(2) h_q is the identity map if h is the identity map, $q \geq 0$

(3) $g_q \circ h_q = (g \circ h)_q$ for all $q \geq 0$, if $g: \mathcal{B} \to \mathcal{C}$ is a second sheaf mor.

(c) For each short exact seq of sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

there is a grp homo

$$\delta^q : H^q(X, \mathcal{C}) \to H^{q+1}(X, \mathcal{A}) \text{ for } \forall q \geq 0 \text{ s.t.}$$

(1) The induced seq

$$0 \to H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \to H^0(X, \mathcal{C}) \xrightarrow{\delta^0} H^1(X, \mathcal{A}) \to \cdots$$
$$\to H^q(X, \mathcal{A}) \to H^q(X, \mathcal{B}) \to H^q(X, \mathcal{C}) \xrightarrow{\delta^q} H^{q+1}(X, \mathcal{A}) \to \cdots$$

is exact

(2) A commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A} \arrow[r] \arrow[d] & \mathcal{B} \arrow[r] \arrow[d] & \mathcal{C} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{A}' \arrow[r] & \mathcal{B}' \arrow[r] & \mathcal{C}' \arrow[r] & 0 \end{tikzcd}$$

induces a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & H^0(X, \mathcal{A}) \arrow[r] \arrow[d] & H^0(X, \mathcal{B}) \arrow[r] \arrow[d] & H^0(X, \mathcal{C}) \arrow[r] \arrow[d] & H^1(X, \mathcal{A}) \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & H^0(X, \mathcal{A}') \arrow[r] & H^0(X, \mathcal{B}') \arrow[r] & H^0(X, \mathcal{C}') \arrow[r] & H^1(X, \mathcal{A}') \arrow[r] & \cdots \end{tikzcd}$$

Pf: (a) (1) Consider resolution $0 \to \mathcal{S} \to \mathcal{C}^0(\mathcal{S}) \to \mathcal{C}^1(\mathcal{S}) \to \cdots$

Take sections. we've known it's exact at $\Gamma(X, \mathcal{S})$ and $\mathcal{C}^0(\mathcal{S})(X) =: \mathcal{C}^0(X, \mathcal{S})$

$$0 \to \Gamma(X, \mathcal{S}) \xrightarrow{i} \mathcal{C}^0(X, \mathcal{S}) \xrightarrow{\delta^0} \mathcal{C}^1(X, \mathcal{S}) \to \cdots$$

(Note that we shall truncate $\Gamma(X, \mathcal{S})$ to compute $H^0(X, \mathcal{S})$)

$$H^0(X,\mathcal{S}) = \ker\delta^0/0 = \ker\delta^0 \underset{\substack{\uparrow \\ \text{exact at} \\ C^0(X,\mathcal{S})}}{=} \operatorname{Im} 2 \underset{\substack{\uparrow \\ \text{exact at} \\ \Gamma(X,\mathcal{S})}}{=} \Gamma(X,\mathcal{S})$$

(a)(2) $\mathcal{S}$ is soft, so the canonical resolution of soft sheaf is an exact seq of soft sheaves $0 \to \mathcal{S} \to C^0(\mathcal{S}) \to C^1(\mathcal{S}) \to \cdots$

Hence by prop we have $0 \to \Gamma(X,\mathcal{S}) \to C^0(\mathcal{S})(X) \to C^1(\mathcal{S})(X) \to \cdots$ is also exact. Therefore $H^q(X,\mathcal{S}) = 0$ for $q > 0$.

(b) & (c). Note that for $h: \mathcal{A} \to \mathcal{B}$, it induces naturally a cochain complex map $h^*: C^*(\mathcal{A}) \longrightarrow C^*(\mathcal{B})$.

Recall that $C^0(\mathcal{A})(U) = \{ f: U \to \tilde{\mathcal{A}} \mid \pi f = 1_U \}$ be sheaf of discontinous sections of $\tilde{\mathcal{S}}$ over X.

So we define $h^0: C^0(\mathcal{A}) \longrightarrow C^0(\mathcal{B})$ by $h^0_U: C^0(\mathcal{A})(U) \to C^0(\mathcal{B})(U)$

$$\underset{\text{where } s \in \mathcal{A}(U)}{\begin{bmatrix} \tilde{s}: U \to \tilde{\mathcal{A}} \\ x \mapsto s_x \end{bmatrix}} \longmapsto \underset{h_U s \in \mathcal{B}(U)}{\begin{bmatrix} U \to \tilde{\mathcal{B}} \\ x \mapsto (h_U s)_x \end{bmatrix}}$$

There is a injective sheaf mor $f: \mathcal{A} \to C^0(\mathcal{A})$ by $f_U: \mathcal{A}(U) \to C^0(\mathcal{A})(U)$, $s \longmapsto \begin{bmatrix} \tilde{s}: U \to \tilde{\mathcal{A}} \\ x \mapsto s_x \end{bmatrix}$. We view $\mathcal{A}$ as subsheaf of $C^0(\mathcal{A})$ and $\mathcal{B}$ a subsheaf of $C^0(\mathcal{B})$. Note that $h^0_U(\mathcal{A}(U)) \subseteq \mathcal{B}(U)$ ($h^0_U(s) = h_U s$) so h^0 induces a mor $h^0: C^0(\mathcal{A})/\mathcal{A} \to C^0(\mathcal{B})/\mathcal{B}$. By definition, $C^0(\mathcal{A})/\mathcal{A} = \mathcal{F}^1(\mathcal{A})$. Hence $h^0: \mathcal{F}^1(\mathcal{A}) \to \mathcal{F}^1(\mathcal{B})$. Repeat above steps, we have a mor $h^1: C^0(\mathcal{F}^1(\mathcal{A})) \longrightarrow C^0(\mathcal{F}^1(\mathcal{B}))$ which is, by definition, $h^1: C^1(\mathcal{A}) \longrightarrow C^1(\mathcal{B})$. Then we have $h^1: C^1(\mathcal{A})/_{\mathcal{F}^1(\mathcal{A})} \to C^1(\mathcal{B})/\mathcal{F}^1(\mathcal{B})$, which is, by def,

$h^1: \mathcal{F}^2(\mathcal{A}) \longrightarrow \mathcal{F}^2(\mathcal{B})$. Then $h^2: \underset{C^2(\mathcal{A})}{\underset{\shortparallel}{C^0(\mathcal{F}^2(\mathcal{A}))}} \longrightarrow \underset{C^2(\mathcal{B})}{\underset{\shortparallel}{C^0(\mathcal{F}^2(\mathcal{B}))}}$

...... Finally, we have $h^*: C^*(\mathcal{A}) \to C^*(\mathcal{B})$.

Since $H^q(X,\mathcal{A}) = H^q(C^*(\mathcal{A}))$, thm (b)(1)(2)(3) are conclusions in Hatcher's alg. topo.

Given $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$, we have $0 \to C^*(\mathcal{A}) \to C^*(\mathcal{B}) \to C^*(\mathcal{C}) \to 0$, then (c) are conclusions in Hatcher's alg topo

[Rmk] These properties can be used as axioms for cohomology theory, and one can prove existence and uniqueness of a cohomology theory with thoes axioms.

The rest part we want to focus on the computation.

[Def] A resolution of a sheaf $\mathcal{S}$ over a space X

$$0 \to \mathcal{S} \to \mathcal{A}^*$$

is called acyclic if $H^q(X, \mathcal{A}^p) = 0$ for $\forall q > 0$ and $p \geq 0$

[Exp] By above thm, soft resolution of a sheaf is acyclic.

Acyclic resolution of sheaves give us one way of computing the cohomology grps of a sheaf by following thm

[Thm] (Abstract de Rham thm) Let $\mathcal{S}$ be a sheaf over X and let $0 \to \mathcal{S} \to \mathcal{A}^*$ be a resolution of $\mathcal{S}$. Then there is a natural homo $\gamma^p : H^p(\Gamma(X, \mathcal{A}^*)) \to H^p(X, \mathcal{S})$.
Moreover, if $0 \to \mathcal{S} \to \mathcal{A}^*$ is acyclic, γ^p is an iso.

pf:

- Construct $\gamma^p : H^p(\Gamma(X, \mathcal{A}^*)) \longrightarrow H^p(X, \mathcal{S})$

Common trick: Spliting a long exact seq to short exact seq.

$0 \to \mathcal{A}^0 \xrightarrow{i^0} \mathcal{A}^1 \xrightarrow{i^1} \mathcal{A}^2 \xrightarrow{i^2} \cdots$ Let $\mathcal{K}^p = \ker(\mathcal{A}^p \to \mathcal{A}^{p+1}) = \mathrm{Im}(\mathcal{A}^{p-1} \to \mathcal{A}^p)$

$$\mathcal{A}^0 \xrightarrow{i^0} \mathcal{K}^1 \hookrightarrow \mathcal{A}^1 \xrightarrow{i^1} \mathcal{K}^2 \hookrightarrow \mathcal{A}^2 \xrightarrow{i^2} \mathcal{K}^3 \hookrightarrow \cdots$$

Then we have short exact seq $0 \to \mathcal{K}^p \xrightarrow{i} \mathcal{A}^p \xrightarrow{i^p} \mathcal{K}^{p+1} \to 0$.

With S.E.S., we have L.E.S.:

$0 \to H^0(X, \mathcal{K}^{p-1}) \to H^0(X, \mathcal{A}^{p-1}) \to H^0(X, \mathcal{K}^p) \xrightarrow{\delta^0} H^1(X, \mathcal{K}^{p+1}) \to \cdots$

With resolution $0 \to \mathcal{S} \to \mathcal{A}^*$, we have

$$H^p(\Gamma(X, \mathcal{A}^*)) = \frac{\ker(\Gamma(X, \mathcal{A}^p) \to \Gamma(X, \mathcal{A}^{p+1}))}{\mathrm{Im}(\Gamma(X, \mathcal{A}^{p-1}) \to \Gamma(X, \mathcal{A}^p))}$$

$0 \to \mathcal{K}^p \to \mathcal{A}^p \to \mathcal{K}^{p+1} \subseteq \mathcal{A}^{p+1} \to 0$ exact
so $0 \to \Gamma(X, \mathcal{K}^p) \to \Gamma(X, \mathcal{A}^p) \to \Gamma(X, \mathcal{K}^{p+1}) \to 0$ exact at first two terms. Hence
$\ker(\Gamma(X, \mathcal{A}^p) \to \Gamma(X, \mathcal{A}^{p+1}))$
$= \ker(\Gamma(X, \mathcal{A}^p) \to \Gamma(X, \mathcal{K}^{p+1}))$
$= \Gamma(X, \mathcal{K}^p)$ ← exact at $\Gamma(X, \mathcal{A}^p)$

$$= \frac{\Gamma(X, \mathcal{K}^p)}{\mathrm{Im}(\Gamma(X, \mathcal{A}^{p-1}) \to \Gamma(X, \mathcal{A}^p))}$$

Consider δ^0 in L.E.S. $\delta^0 : \underset{\Gamma(X,\mathcal{K}^p)}{\overset{\shortparallel}{H^0(X,\mathcal{K}^p)}} \longrightarrow H^1(X,\mathcal{K}^{p+1})$

It induces $\gamma_1^p : \underset{(\Gamma(X;\mathcal{K}^p)/\ldots)}{\overset{\shortparallel}{H^p(\Gamma(X,\mathcal{A}^*))}} \longrightarrow H^1(X,\mathcal{K}^{p+1})$.

If the resolution is acyclic, $H^1(X,\mathcal{A}^{p-1})=0$. So in L.E.S. δ^0 is surj and thus γ_1^p is surj. γ_1^p is obviously inj, hence it's iso.

Similarly, consider exact seq $0\to \mathcal{K}^{p-r}\to \mathcal{A}^{p-r}\to \mathcal{K}^{p-r+1}\to 0$ we obtain $\gamma_r^p : H^{r-1}(X,\mathcal{K}^{p-r+1})\longrightarrow H^r(X,\mathcal{K}^{p-r})$ (iso when acyclic)

We define $\gamma_p = \gamma_p^p \circ \gamma_{p-1}^p \circ \cdots \gamma_2^p \circ \gamma_1^p : H^p(\Gamma(X,\mathcal{A}^*)) \to \underset{H^p(X,S)}{\overset{\shortparallel}{H^p(X,\mathcal{K}^0)}}$

which is iso when resolution is acyclic.

$\square$

[Rmk] In the proof we only use cohomology axiom and do not use sheaf property. That's an evidence for axioms are complement.

[Coro] Suppose
$$\begin{array}{ccccc} 0 \to & S & \to & \mathcal{A}^* \\ & \downarrow f & & \downarrow g \\ 0\to & \mathcal{J} & \to & \mathcal{B}^* \end{array}$$
is a homo of resolutions of sheaves.

Then there is an induced homo $H^p(\Gamma(X,\mathcal{A}^*)) \xrightarrow{g_p} H^p(\Gamma(X,\mathcal{B}^*))$ which is, moreover, an isomorphism if f is an iso of sheaves and the resolutions are both acyclic.

pf: Since $H^p(\Gamma(X,-)) \longrightarrow H^p(X,-)$ is natural, we have commutative diagram
$$\begin{array}{ccc} H^p(\Gamma(X,\mathcal{A}^*)) & \xrightarrow{\gamma_A^p} & H^p(X,S) \\ \downarrow g_p & & \downarrow f_p \\ H^p(\Gamma(X,\mathcal{B}^*)) & \xrightarrow[\gamma_B^p]{} & H^p(X,S) \end{array}$$
(g_p is induced)

When f is iso, f_p is iso.
When resolutions acyclic, γ_A^p and γ_B^p are iso. $\Big\}\xRightarrow{\text{diagram commutes}}$ g_p is iso.

$\square$

[Lemma] Let $\mathcal{R}$ be a soft sheaf of ring and $\mathcal{M}$ is a sheaf of $\mathcal{R}$-modules. Then $\mathcal{M}$ is a soft sheaf.

pf: Assume K a closed subset of X. Let $s \in \mathcal{M}(K)$. $\exists$ open $U \supseteq K$ and $\bar{s} \in \mathcal{M}(U)$ s.t. $r^U_K \bar{s} = s$. (property of direct limit) Let $\rho \in \Gamma(K \cup (X-U), \mathcal{R})$ by setting $\rho \equiv \begin{cases} 1 & \text{on } K \\ 0 & \text{on } X-U \end{cases}$. Since $\mathcal{R}$ is soft, there exists $\bar{\rho} \in \Gamma(X, \mathcal{R})$ with $r^X_{K\cup(X-U)} \bar{\rho} = \rho$. $\mathcal{M}$ is a sheaf of $\mathcal{R}$-module, so $\bar{\rho} \cdot \bar{s} \in \mathcal{M}(X)$. $r^X_K \bar{\rho} \cdot \bar{s} = \rho \cdot r^X_K \bar{s} = \rho \cdot s \underset{\rho \equiv 1 \text{ on } K}{=} s$.

compatness of restriction and module action

$$\begin{tikzcd}
\mathcal{M}(X) \arrow[r, "\bar{\rho}"] \arrow[d, "r^V_K"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{M}(X) \arrow[d, "r^X_K"] \\
\mathcal{M}(K) \arrow[r, "\rho \circ -"] & \mathcal{M}(K)
\end{tikzcd}$$

[Thm] (de Rham) Let X be a differentiable mf. Then the natural mapping $I : H^p(\mathcal{E}^*(X)) \longrightarrow H^p(S^*_\infty(X, \mathbb{R}))$ induced by $\mathcal{E}^*(X) \longrightarrow S^*_\infty(X, \mathbb{R})$, $\varphi \longmapsto \int_c \varphi$ is an iso.

C^∞ singular cochains with coefficients in $\mathbb{R}$.

pf: Consider resolutions of $\mathbb{R}$ in one of our examples.

$$\begin{tikzcd}
& & \mathcal{E}^* \\
0 \arrow[r] & \mathbb{R} \arrow[ur, "i"] \arrow[dr, "i"'] & \\
& & S^*_\infty
\end{tikzcd}$$

Claim: $\mathcal{E}^*$ and S^*_∞ are both soft.

If the claim is true, we have iso $H^p(\mathcal{E}^*(X)) \to H^p(S^*_\infty(X, \mathbb{R}))$ by above corollary.

- $\mathcal{E}^*$ is fine, so $\mathcal{E}^*$ is soft.
- Show S^*_∞ is soft. By cup product, we find that S^*_∞ is an S^0_∞-module. Claim: S^0_∞ is soft. If this claim is true, S^*_∞ is soft as a module of soft sheaf. Then we show S^0_∞ is soft: $S^0_\infty(U) = \{f : S_0(U) \to \mathbb{R} \mid f \text{ is } C^\infty\} = \{f : U \to \mathbb{R} \mid f\ C^\infty\} = C_\infty(U, \mathbb{R})$. So S^0_∞ is soft. (A bit different from Gtm 65, I guess this is what Gtm 65 mean) $\square$

[Thm] (Dolbeault) Let X be a complex mf. Then

$$H^q(X, \Omega^p) \cong \frac{\ker(\mathcal{E}^{p,q}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,q+1}(X))}{\mathrm{Im}(\mathcal{E}^{p,q-1}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,q}(X))}$$

pf: Consider the resolution of soft sheaves:

$$0 \to \Omega^p \xrightarrow{i} \mathcal{E}^{p,0} \xrightarrow{\bar\partial} \mathcal{E}^{p,1} \xrightarrow{\bar\partial} \cdots \longrightarrow \mathcal{E}^{p,n} \to 0$$

Then by abstract de Rham thm, we have

$$H^q(X, \Omega^p) \cong H^q(\Gamma(X, \mathcal{E}^{p,*}))$$

$$= \frac{\ker(\mathcal{E}^{p,q}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,q+1}(X))}{\mathrm{Im}(\mathcal{E}^{p,q-1}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,q}(X))}$$

$H^q(\Gamma(X, \mathcal{E}^{p,*}))$ is the q-th homology grp of a chain complex

$$\cdots \longrightarrow \mathcal{E}^{p,q-1}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,q}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,q+1}(X) \longrightarrow \cdots \qquad \square$$

Next, we let bundles play a role in de Rham thm.

[Def] Let $\mathcal{M}$ and $\mathcal{N}$ be sheaves of modules over a sheaf of commutative rings $\mathcal{R}$. Let $\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N}$ denote the sheaf generated by presheaf $U \to \mathcal{M}(U) \otimes_{\mathcal{R}} \mathcal{N}(U)$ and we call sheaf $\mathcal{M} \otimes_{\mathcal{R}} \mathcal{N}$ the tensor product of $\mathcal{M}$ and $\mathcal{N}$.

[Rmk] presheaf $U \to \mathcal{M} \otimes_{\mathcal{R}} \mathcal{N}$ is not a sheaf. We provide a contraexample here. Let $E \to X$ be a holomorphic vector bundle with no nontrivial global holomorphic sections.

We have sheaf $\mathcal{O}(E)$ by $\mathcal{O}(E)(U) = \{$all holo sections of E over $U\}$

We have sheaf $\mathcal{E}$ by $\mathcal{E}(U) = \{$all differential functions on $U\}$

$\mathcal{O}(E)$ and $\mathcal{E}$ are sheaves of $\mathcal{O}$-module where $\mathcal{O}$ is the structure sheaf setting by $\mathcal{O}(U) = \{$all holo funs on $U\}$

Let $\{U_j\}$ be the sets of trivializing cover of X. We have $(\mathcal{O}(E)\otimes_{\mathcal{O}}\mathcal{E})(X) = \mathcal{O}(E)(X)\otimes_{\mathcal{O}(X)}\mathcal{E}(X) = 0$ (since there are no nontrivial global holomorphic sections, $\mathcal{O}(E)(X)=0$.)

On the other side, $(\mathcal{O}(E)\otimes_{\mathcal{O}}\mathcal{E})(U_j) = \mathcal{O}(E)(U_j)\otimes_{\mathcal{O}(U_j)}\mathcal{E}(U_j) \cong \mathcal{E}(E)(U_j) \neq 0$. Thus we have nontrivial patch of sections, if $\mathcal{O}(E)\otimes_{\mathcal{O}}\mathcal{E}$ is a sheaf we can glue patches of nontrivial sections to obtain a global nontrivial section, but we find there are no global nontrivial section since $(\mathcal{O}(E)\otimes_{\mathcal{O}}\mathcal{E})(X)=0$.

Hence it's not a sheaf. (We define $\mathcal{O}(E)\otimes_{\mathcal{O}}\mathcal{E}$ the presheaf here).

[prop] $(\mathcal{M}\otimes_{\mathcal{R}}\eta)_x = \mathcal{M}_x\otimes_{\mathcal{R}_x}\eta_x$

pf: Denote $\mathcal{H}$ the presheaf $U\to \mathcal{M}(U)\otimes_{\mathcal{R}(U)}\eta(U)$.

Sheafification doesn't change stalks, so $(\mathcal{M}\otimes_{\mathcal{R}}\eta)_x = \mathcal{H}_x$

Hence it suffices to show $\mathcal{H}_x = \mathcal{M}_x\otimes_{\mathcal{R}_x}\eta_x$

By concrete construction of stalks, $\mathcal{H}_x = \coprod \mathcal{H}(U)/\sim$

$$= \{[(U,f)] \mid U \text{ open in } X,\ f\in\mathcal{H}(U)=\mathcal{M}(U)\otimes_{\mathcal{R}(U)}\eta(U)\}$$

By construction of tensor product

$$= \{[(U,\textstyle\sum a_i u_i\otimes v_i)] \mid U\overset{\text{open}}{\subseteq} X,\ a_i\in\mathcal{R}(U),\ u_i\in\mathcal{M}(U),\ v_i\in\eta(U)\}$$

$$\mathcal{M}_x\otimes_{\mathcal{R}_x}\eta_x = \left\{\sum_i [(U,a_i)][(U,u_i)]\otimes[(U,v_i)] \;\middle|\; \begin{array}{l}[(U,a_i)]\in\mathcal{R}_x\\ [(U,u_i)]\in\mathcal{M}_x\\ [(U,v_i)]\in\eta_x\end{array}\right\}$$

we can always change representative elements as this form.

$$= \left\{[(U,\textstyle\sum a_i u_i\otimes v_i)] \;\middle|\; \begin{array}{l}U\overset{\text{open}}{\subseteq} X,\ a_i\in\mathcal{R}(U)\\ u_i\in\mathcal{M}(U),\ v_i\in\eta(U)\end{array}\right\}$$

$$= \mathcal{H}_x.$$

□

[Lemma] If $\mathcal{J}$ is a locally free sheaf of $\mathcal{R}$-modules and

$$0 \to \mathcal{A}' \to \mathcal{A} \to \mathcal{A}'' \to 0$$ is a short exact seq of

$\mathcal{R}$-modules, then

$$0 \to \mathcal{A}' \otimes_{\mathcal{R}} \mathcal{J} \to \mathcal{A} \otimes_{\mathcal{R}} \mathcal{J} \to \mathcal{A}'' \otimes_{\mathcal{R}} \mathcal{J} \to 0$$

is also exact.

pf: For any $x \in X$

$$0 \to (\mathcal{A}' \otimes_{\mathcal{R}} \mathcal{J})_x = \mathcal{A}'_x \otimes_{\mathcal{R}_x} \mathcal{J}_x \to \mathcal{A}_x \otimes_{\mathcal{R}} \mathcal{J}_x \to \mathcal{A}''_x \otimes_{\mathcal{R}_x} \mathcal{J}_x \to 0$$

is exact, since exact seq tensor free module is also exact by basic algebra. $\square$

Recall that there is a resolution of sheaves of $\mathcal{O}$-modules over a complex m.f. X:

$$0 \to \Omega^p \to \mathcal{E}^{p,0} \xrightarrow{\bar\partial} \mathcal{E}^{p,1} \xrightarrow{\bar\partial} \cdots \to \mathcal{E}^{p,n} \to 0$$

If X admits a holomorphic bundle E, we have sheaf $\mathcal{O}(E)$. We've proved $\mathcal{O}(E)$ is locally free in the thm illustrating correspondence of $\mathcal{S}$-bundles and locally free $\mathcal{S}$-sections. Exact seq tensor locally free sheaf is also exact, i.e.

$$0 \to \Omega^p \otimes_{\mathcal{O}} \mathcal{O}(E) \to \mathcal{E}^{p,0} \otimes_{\mathcal{O}} \mathcal{O}(E) \xrightarrow{\bar\partial \otimes 1} \cdots \xrightarrow{\bar\partial \otimes 1} \mathcal{E}^{p,n} \otimes_{\mathcal{O}} \mathcal{O}(E) \to 0$$

is an exact seq.

[Prop] $\Omega^p \otimes_{\mathcal{O}} \mathcal{O}(E) \cong \mathcal{O}(\wedge^p T^*(X) \otimes_{\mathbb{C}} E)$

pf: We should use two facts: 1. E, F be bundles over mf M. Γ be section sheaf, we have $\Gamma(E \otimes F) = \Gamma(E) \otimes_{C^\infty(M)} \Gamma(F)$, more details: https://math.stackexchange.com/questions/1857939/sections-of-tensor-bundle-are-tensor-product-of-sections

2. Recall that $\Omega^p = \ker(\mathcal{E}^{p,0} \xrightarrow{\bar\partial} \mathcal{E}^{p,1})$, actually it's the sheaf of holomorphic differential forms of type $(p,0)$, i.e., in local coord, $\varphi \in \Omega^p(U)$ iff $\varphi = \sum_{|I|=p}' \varphi_I dz^I$, $\varphi_I \in \mathcal{O}(U)$. So $\Omega^p = \mathcal{O}(\wedge^p T^*(X))$.

With those facts, we have $\mathcal{O}(\Lambda^p T^*(X) \otimes_{\mathbb{C}} E) \cong \mathcal{O}(\Lambda^p T^*(X)) \otimes_{\mathcal{O}} \mathcal{O}(E)$
$\cong \Omega^p \otimes_{\mathcal{O}} \mathcal{O}(E)$.

[Prop] $\mathcal{E}^{p,q} \otimes_{\mathcal{O}} \mathcal{O}(E) \cong \mathcal{E}(\Lambda^{p,q} T^*(X) \otimes_{\mathbb{C}} E)$.

Pf: $\mathcal{E}(\Lambda^{p,q} T^*(X) \otimes_{\mathbb{C}} E) = \mathcal{E}(\Lambda^{p,q} T^*(X)) \otimes_{\mathcal{E}} \mathcal{E}(E)$

$\mathcal{E}^{p,q} := \mathcal{E}(\Lambda^{p,q} T^*(X))$ $\quad = \mathcal{E}(\Lambda^{p,q} T^*(X)) \otimes_{\mathcal{O}} \mathcal{O}(E)$

$= \mathcal{E}^{p,q} \otimes_{\mathbb{C}} \mathcal{O}(E)$

section的性质由性质差的决定 differentiable与holo放一块最终还是differentiable

[Rmk] In "$\Delta \otimes_{\mathcal{F}} \square$", $\Delta, \square$ are $\mathcal{F}$-modules.

[Prop] $\mathcal{O}(E) \otimes_{\mathcal{O}} \mathcal{E} = \mathcal{E}(E)$

$$\mathcal{E}(E) = \mathcal{E}(E) \otimes_{\mathcal{E}} \mathcal{E} = \mathcal{O}(E) \otimes_{\mathcal{O}} \mathcal{E}$$

[Def] $\mathcal{O}(X, \Lambda^p T^*(X) \otimes_{\mathbb{C}} E)$ is called <u>the (global) holomorphic p-forms on X with coefficients in E</u>, denoted by $\Omega^p(X, E)$. We denote the sheaf of <u>holomorphic p-forms with coefficients in E</u> by $\Omega^p(E)$. Let $\mathcal{E}^{p,q}(X, E) := \mathcal{E}(X, \Lambda^{p,q} T^*(X) \otimes_{\mathbb{C}} E)$ be <u>the differentiable (p,q)-forms on X with coefficients in E</u>.

[Rmk] $\underset{\underbrace{\Omega^p(E)}_{\text{sheaf}}\underbrace{(X)}_{\text{valued at } X}}{\underset{\parallel}{\Omega^p(X, E)}} = \underset{\text{means } \mathcal{O}\text{-sections}}{\underline{\underline{\mathcal{O}}}}(X, \Lambda^p T^*(X) \otimes_{\mathbb{C}} E) = \underbrace{\mathcal{O}(\Lambda^p T^*(X) \otimes_{\mathbb{C}} E)}_{\text{sheaf}} \underbrace{(X)}_{\text{valued at } X}$

$$= \mathcal{O}(\Lambda^p T^*(X)) \otimes_{\mathcal{O}} \mathcal{O}(E) \, (X)$$

p-forms $\quad$ coeffs in E

This is the sheafification of presheaf $U \to \Omega^p(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(E)(U)$ $\longrightarrow$ $= \Omega^p \otimes_{\mathcal{O}} \mathcal{O}(E) \, (X)$

$$\neq \Omega^p(X) \otimes_{\mathcal{O}} \mathcal{O}(E)(X).$$

So $\Omega^p(E) = \Omega^p \otimes_{\mathcal{O}} \mathcal{O}(E)$.

Similarly $\mathcal{E}^{p,q}(X, E) = \underbrace{\mathcal{E}^{p,q}(E)}_{\text{sheaf}} \underbrace{(X)}_{\text{valued at } X} = \mathcal{E}^{p,q} \otimes_{\mathcal{O}} \mathcal{O}(E)(X)$.

$\mathcal{E}^{p,q}(E) = \mathcal{E}^{p,q} \otimes_{\mathcal{O}} \mathcal{O}(E)$.

Then the long exact seq can be written as

$$0 \to \Omega^p(E) \to \mathcal{E}^{p,0}(E) \to \mathcal{E}^{p,1}(E) \xrightarrow{\bar\partial_E} \cdots \xrightarrow{\bar\partial_E} \mathcal{E}^{p,n}(E) \to 0$$

where $\bar\partial = \bar\partial \otimes 1$. It's exact and $\mathcal{E}^{p,q}(E)$ are fine sheaves, so we have following generalization of Dolbeault's thm.

[Thm] (Dolbeault's thm) Let X be a complex m.f. and let $E \to X$ be a holomorphic vector bundle. Then

$$H^q(X, \Omega^p(E)) \cong \frac{\ker(\mathcal{E}^{p,q}(X,E) \xrightarrow{\bar\partial_E} \mathcal{E}^{p,q+1}(X,E))}{\operatorname{Im}(\mathcal{E}^{p,q-1}(X,E) \to \mathcal{E}^{p,q}(X,E))}$$

Čech cohomology with coefficients in a sheaf

This section has similar process as in defining singular homology.

Let X be a topo space, $\mathcal{F}$ be a sheaf of ab grps on X. Let $\mathcal{U}$ be a covering of X by open sets.

[Def] (q-simplex) • A q-simplex σ is an ordered collection of $q+1$ sets of the covering $\mathcal{U}$ with nonempty intersection, i.e., $\sigma = (U_0, \cdots, U_q)$ with $\bigcap_{i=0}^{q} U_i \neq \phi$.

• We call the set $\bigcap_{U_i \in \sigma} U_i =: |\sigma|$ the support of the simplex σ.

• A q-cochain of $\mathcal{U}$ with coefficients in $\mathcal{F}$ is a mapping f which associates to each q-simplex σ a $f(\sigma) \in \mathcal{F}(|\sigma|)$.

• Let $C^q(\mathcal{U}, \mathcal{F})$ denote the set of q-cochains, which is an abelian grp.

• Define coboundary operator $\delta : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})$ by $\delta f(\sigma) = \sum_{i=0}^{q+1} (-1)^i r^{|\sigma_i|}_{|\sigma|} f(\sigma_i)$ where $f \in C^q(\mathcal{U}, \mathcal{F})$, $\sigma_i = (U_0, \cdots, \widehat{U_i}, \cdots, U_{q+1})$ and $r^{|\sigma_i|}_{|\sigma|}$ is the sheaf restriction.

[Prop] 1. δ is a grp homo

2. $\delta^2 = 0$

3. We have cochain complex

$$C^*(\mathcal{U}, \mathcal{S}) := [C^0(\mathcal{U}, \mathcal{S}) \to \cdots \to C^q(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta} C^{q+1}(\mathcal{U}, \mathcal{S}) \to \cdots]$$

[Def] Cohomology of cochain complex $C^*(\mathcal{U}, \mathcal{S})$ is the Čech cohomology. $Z^q(\mathcal{U}, \mathcal{S}) := \ker \delta$, $B^q(\mathcal{U}, \mathcal{S}) := \operatorname{Im} \delta$, and

$$H^q(\mathcal{U}, \mathcal{S}) := H^q(C^*(\mathcal{U}, \mathcal{S})) = Z^q(\mathcal{U}, \mathcal{S}) / B^q(\mathcal{U}, \mathcal{S})$$

[prop] If $\mathfrak{M}$ is a refinement of $\mathcal{U}$, then there is a natural grp homo $\mu^{\mathcal{U}}_{\mathfrak{M}} : H^q(\mathcal{U}, \mathcal{S}) \longrightarrow H^q(\mathfrak{M}, \mathcal{S})$ and

$$\boxed{\varinjlim_{\mathcal{U}} H^q(\mathcal{U}, \mathcal{S}) \cong H^q(X, \mathcal{S})}$$ ← We can represent $H^*(X, \mathcal{S})$ by Čech cohomology.

[prop] If $\mathcal{U}$ is a covering s.t. $H^q(|\sigma|, \mathcal{S}) = 0$ for $q \geq 1$ and all simplices σ in $\mathcal{U}$, then $H^q(X, \mathcal{S}) \cong H^q(\mathcal{U}, \mathcal{S})$ for all $q \geq 0$ and we call $\mathcal{U}$ a Leray cover.

[prop] If X is paracompact, $\mathcal{U}$ is locally finite covering, and $\mathcal{S}$ is a fine sheaf over X, then $H^q(\mathcal{U}, \mathcal{S}) = 0$ for $q > 0$