When do we have k-spectral bundles?

Ref: https://arxiv.org/abs/math/0304281

[Notation]: $E(\lambda, M)$: eigenspace corresponding to eigenvalue λ of matrix M.

k-spectral bundles and examples

[Def] vector bundle is a fiber bundle whose fiber is vector space and trivialization maps are linear isomorphisms.

[Def] Let B be any space and V an n-dim K-Vector space. We call $\Phi: B \to \mathrm{Hom}(V,V)$ a parametrization by B. In particular, if we've found a basis for V, we have $\Phi: B \to M_n(k)$.

[Exp] $\Phi: \mathbb{R}^2 \to M_2(\mathbb{R})$ is a parametrization.

$$t \mapsto \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

[Def] k-spectral bundle corresponding to a parametrization $\Phi: B \to M_n(k)$ is a vector bundle $\pi: H \to B$ over B satisfying that each fiber $\pi^{-1}(b)$ is a k-dimensional vector subspace of $E(\lambda, \Phi(b))$ for some eigenvalue λ of $\Phi(b)$.

[Rmk] $\pi^{-1}(b) \subseteq E(\lambda, \Phi(b))$ for some λ suggesting that $\dim E(\lambda, \Phi(b)) \geq k$.

The following example which we curious about are special cases of k-spectral bundle.

[Exp] Let $\Phi : B \longrightarrow M_n(k)$ is a parametrization. If there exists t-th eigenvalue of $\Phi(b)$ for $\forall b \in B$ s.t.

$$\dim E(\lambda_t, \Phi(b)) = k \text{ and } \pi : \begin{array}{rcl} \bigsqcup_{b\in B} E(\lambda_t, \Phi(b)) & \longrightarrow & B \\ \vec{v} \in E(\lambda_t, \Phi(b)) & \longmapsto & b \end{array}$$

is a vector bundle, then π is a k-spectral bundle.

[Rmk]: We can see drawbacks of k-spectral bundle (or vector bundle) (1) we require each fiber (a vector space) has same dimension which is quite awful. Because dimension of eigenspaces may varied when b runs over B and we may choose the minimum dimension of eigenspaces to be our k.

(2) To form a k-spectral bundle, at most of time we just consider the eigenspace of one eigenvalue. However, Higgs bundles simultanious consider all eigenvalues of $\Phi(b)$ at one $b \in B$.

Although k-spectral bundle is not very useful, it's much easier than Higgs bundles and has a striking criterian to judge whether a parametrization admits a k-spectral bundle —— the following we'll study existence of k-spectral bundle. We first provide two examples with one admit and the other doesn't admit k-spectral bundle.

[Exp] $\Phi : \mathbb{R} \longrightarrow M_2(\mathbb{R})$

$$t \longmapsto \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \Rightarrow \Phi(t)$$

eigenvalues	eigenspaces
1	$\mathbb{R}^2$ $t=0$ $\mathrm{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $t\neq 0$

Then we have a 1-spectral bundle (a trivial line bundle) (vector bundle with dimension 1)

$\pi : \mathbb{R} \times \mathrm{span}\binom{1}{0} \longrightarrow \mathbb{R}$, $(t, v) \longmapsto t$

[Exp] Let $f(t)$ be a continous function over $\mathbb{R}$

with $\begin{cases} f(t) > 0, & t > 0 \\ f(t) = 0, & t \leq 0 \end{cases}$.

Let parametrization $\Phi : \mathbb{R} \longrightarrow M_2(\mathbb{R})$ is given by $\Phi(t) = \begin{bmatrix} 1 & f(t) \\ f(-t) & 1 \end{bmatrix}$.

$\Phi(t)$	eigenvalues	eigenspaces	
		span $\begin{bmatrix}1\\0\end{bmatrix}$	$t>0$
	1	$\mathbb{R}^2$	$t=0$
		span $\begin{bmatrix}0\\1\end{bmatrix}$	$t<0$

If Φ admits a 1-spectral bundle $\pi : H \to \mathbb{R}$, we must have $\pi^{-1}(t) = \begin{cases} \mathrm{span}\begin{bmatrix}1\\0\end{bmatrix} & t > 0 \\ \mathrm{span}\begin{bmatrix}0\\1\end{bmatrix} & t < 0 \end{cases}$, so there is no choice of $\pi^{-1}(0)$ such that span$\begin{bmatrix}1\\0\end{bmatrix}$ can continously contract to span $\begin{bmatrix}0\\1\end{bmatrix}$.

[Rmk] degeneracy of eigenspace is an obstruction of admiting a k-spectral bundle.

Judge criterion

Main ideal of this judge criterion is judge whether each part of sections exist.

[Construction] Let $\Phi: B \to \mathrm{Hom}(V,V)$ be a parametrization

Let $L_3 := \{(b,\lambda,W,\vec{v}) \mid (b,\lambda,W,\vec{v}) \in B\times k\times G_{k,n}\times V,$
λ is an eigenvalue of $\Phi(b)$,
W is the eigenspace associated to λ,
$\vec{v}$ is an eigenvector in $W\}$
$\subseteq B\times k\times G_{k,n}\times V$.

Denote projections π_3, π_2, π_1 as following:

$$\begin{array}{ccccccc} B\times k\times G_{k,n}\times V & \xrightarrow{\pi_3} & B\times k\times G_{k,n} & \xrightarrow{\pi_2} & B\times k & \xrightarrow{\pi_1} & B \\ (b,\lambda,W,\vec{v}) & \longmapsto & (b,\lambda,W) & \longmapsto & (b,\lambda) & \longmapsto & b. \end{array}$$

We always consider π_3 restricts to L_3 and write $\pi_3(L_3) =: L_2$, $\pi_2(L_2) := L_1$, we obtain

$$L_3 \xrightarrow{\pi_3} L_2 \xrightarrow{\pi_2} L_1 \xrightarrow{\pi_1} B$$

[Judge criterion]

$$\{k\text{-spectral bundles}\} \xleftrightarrow{1:1} \{\underline{\text{sections}} \text{ of } \pi_2\pi_1 : L_2 \to B\}$$

$(s: B \to L_2$ s.t. $\pi_2\pi_1 s = id)$

pf: Given any k-spectral bundle $\pi: H \to B$.

Then $\begin{array}{rcl} s: B & \longrightarrow & L_2 \subseteq B\times k\times G_{k,n} \\ b & \longmapsto & (b,\lambda,\pi^{-1}(b)) \end{array}$ is a section.

where λ is the eigenvalue associated to k-dimension subeigenspace $\pi^{-1}(b) \subseteq E(\lambda, \Phi(b))$.

Given any section $s: B \to L_2$. Note that we always identify $s(B)$ as B, so to construct k-spectral bundle over B we only need to construct k-spectral bundle over $S(B)$. Since $S(B) \subseteq L_2$, it suffices to construct k-spectral bundle over L_2. The cheapest k-spectral bundle is $\pi_3 : L_3 \to L_2$, $(b, \lambda, W, v) \mapsto (b, \lambda, W)$. Indeed, π_3 is the k-spectral bundle of parametrization

$L_2 \longrightarrow \mathrm{Hom}(V,V)$, $(b,\lambda,W) \mapsto \Phi(b)$. Hence we do a restriction $\pi_3 : L_3' \to S(B) \cong B$ which is a k-spectral bundle over B

[Rmk] $\Phi : B \to \mathrm{Hom}(V, V)$ admits a k-spectral bundle

$$\Updownarrow$$

there exists a section s of $\pi_2\pi_1 : L_2 \to B$.

To check whethere the section exists, we'll introduce a detailed judging method.

Firstly, we need break a section s to two sections.

[Construction] Let's break the section s of $\pi_2\pi_1 : L_2 \to B$.

Let $s_1 := \pi_2 s$ and $s_2 := s\pi_1$.

Since:

$$\begin{cases} s_2 s_1 = s\pi_1\pi_2 s = s \circ id = s \\ \pi_1 s_1 = \pi_1\pi_2 s = id \\ \pi_2 s_2 s_1(b) = \pi_2 s(b) = s_1(b) \Rightarrow \pi_2 s_2|_{\mathrm{Im}\, s_1} = id \end{cases}$$

$$\begin{tikzcd} L_2 \arrow[r, "\pi_2"] & L_1 \arrow[l, bend left, "s_2"] \arrow[r, "\pi_1"] & B \arrow[l, bend left, "s_1"] \arrow[ll, bend right, "s"'] \end{tikzcd}$$

We break s to sections of $L_1 \xrightarrow{\pi_1} B$ and $\pi_2 : L_2' \to s_1(B)$

Secondly, we judge whether s_1, s_2 exists by the following thm.

$$\begin{tikzcd} L_3'' \arrow[r] & L_3' \arrow[r] \arrow[d, "\pi_3"] & L_3 \arrow[r] \arrow[d, "\pi_3"] & B \times \mathbb{K} \times G_{k,n} \times V \arrow[d, "\pi_3"'] \\ s_2 s_1 B \arrow[r] \arrow[u, "s_3"] & L_2' \arrow[r] & L_2 \arrow[r] \arrow[d, "\pi_2"] & B \times \mathbb{K} \times G_{k,n} \arrow[d, "\pi_2"'] \\ & s_1 B \arrow[r] \arrow[u, "s_2"] & L_1 \arrow[r] & B \times \mathbb{K} \arrow[d, "\pi_1"'] \\ & & B \arrow[u, "s_1"] \arrow[r, equal] & B \end{tikzcd}$$

$\Leftarrow$ Picture describe maps s_1, s_2, s_3.

[Thm]

(a) $\{\text{sections } s_1 \text{ of } \pi_1 : L_1 \to B\} \xleftrightarrow{1:1} \{$ conti fun $\lambda : B \to K$ with $\lambda(b)$ an eigenvalue of $\Phi(b)$ and $\dim E(\lambda(b), \Phi(b)) \geq k\}$

(b) $\{\text{sections } s_2 \text{ of } \pi_2 : L_2 \to s_1 B\} \xleftrightarrow{1:1} \{$ conti sections of k-dim subspaces of $E(\lambda(b), \Phi(b))\}$

(c) $\{$nowhere zero sections s_3 of $\pi_3 : L_3'' \to s_2 s_1 B \cong B\}$

$\updownarrow 1:1$

$\{$nowhere zero eigenvector fields for the eigenbundle$\}$

[Rmk] We only need (a), (b).

[Rmk] $\pi : H \to B$ is a k-spectral bundle.

$\pi^{-1}(b) \subseteq E(\lambda(b), \Phi(b))$ where $\lambda : B \to K$ with $\lambda(b)$ a "good" choice of eigenvalues of $\Phi(b)$.

[Exp] $\Phi : \overset{B}{\overset{\shortparallel}{\mathbb{R}}} \longrightarrow M_2(\mathbb{R})$

$t \longmapsto \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \Rightarrow \Phi(t)$

eigenvalues	eigenspaces
1	$\mathbb{R}^2$ $t=0$ $\text{span}\binom{1}{0}$ $t \neq 0$

Judge B has a 1-spectral bundle.

section s_1 exists: there is only one eigenvalue, so $\lambda : B \to K$, $b \mapsto 1$ is a "good" choice of eigenvalues with $\dim(E(\lambda(b), \Phi(b))) = \dim(E(1, \Phi(b))) \geq 1$.

section s_2 exists: $s_1 B \longrightarrow L_2 \subseteq B \times K \times G_{1,2}$

$s_1(b) \longmapsto (b, 1, \text{span}\begin{bmatrix} 1 \\ 0 \end{bmatrix})$

is a continous choice of 1-dim subspaces of $E(\lambda(b)=1, \Phi(b))$.

[Exp] Let $f(t)$ be a continous function over $\mathbb{R}$

with $\begin{cases} f(t) > 0, & t > 0 \\ f(t) = 0, & t \leq 0 \end{cases}$.

Let parametrization $\Phi : \overset{B}{\overset{\shortparallel}{\mathbb{R}}} \longrightarrow M_2(\mathbb{R})$

is given by $\Phi(t) = \begin{bmatrix} 1 & f(t) \\ f(-t) & 1 \end{bmatrix}$.

$\Phi(t)$	eigenvalues	eigenspaces	
		span $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$t>0$
	1	$\mathbb{R}^2$	$t=0$
		span $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$t<0$

section S_1 exists: there is only one eigenvalue

$\lambda : B \to K$, $b \mapsto 1$ is the good choice.

section S_2 doesn't exists: We need to find a continous choice of 1-dim subspaces of $E(1, \Phi(b)) = \begin{cases} \text{span} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & b>0 \\ \mathbb{R}^2 & b=0 \\ \text{span} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & b<0 \end{cases}$

it's impossible.

[Exp] $\Phi : B = \mathbb{R}^3 \longrightarrow M_2(\mathbb{R})$ doesn't admit

$(u, v, w) \longmapsto \begin{pmatrix} u & v \\ v & w \end{pmatrix}$ 1-spectral bundle.

section S_1 exists: $\Phi(b)$ is a symmetric matrix so eigenvalues are real and we can find continous $\lambda : B \to \mathbb{R}$.

section S_2 doesn't exist: Suppose S_2 exists. Then by thm there is a continous choice of 1-dim subspace of $E(\lambda(b), \Phi(b))$ so there is a line bundle over $\mathbb{R}^3$. Since $\mathbb{R}^3$ is contractible, this line bundle over $\mathbb{R}^3$ is also contracitible, hence it's a trivial line bundle over $\mathbb{R}^3$. Let $l = \{(u,0,u) \mid u \in \mathbb{R}\} \subseteq B$ which is the line where eigenvalue is u and eigenspace is $\mathbb{R}^2$. off the line l $\Phi(b)$ has eigenvalue u, w and each associated to a 1-dim eigenspace. The trivial line bundle over $\mathbb{R}^3$ can restrict to a loop close l which is also trivial, leading to a contradiction.

If we consider $\Phi : B' = \mathbb{R}^3 - l \to M_2(\mathbb{R})$, section S_2 exists and admit a nontrivial 1-spectral bundle (line bundle). Real line bundles are classified by Stiefel-Whitney class which lies in $H^1(B'; \mathbb{Z}_2) \cong H^1(S^1; \mathbb{Z}_2) = \mathbb{Z}_2$. So this nontrivial line bundle corresponding to the unique nontrivial element in $\mathbb{Z}_2$.

[Rmk] Complex line bundles are classified by first Chern class, which lies in $H^2(B; \mathbb{Z})$ where B is the base space of the bundle.