How to tell a fiber bundle is trivial?

* Note that even if total space is iso, we can not tell two bundles are iso.(要考虑 bundle iso. 反例见:

https://math.stackexchange.com/questions/1943066/nonisomorphic-vector-bundles-with-diffeomorphic-total-spaces
)

▲ 可以从 long exact seq induced by fiber bundle

$F \to E \to B \rightsquigarrow$ we have L.E.S.

$\pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots \to \pi_1(B)$

$\to \pi_0(F) \to \pi_0(E) \to \pi_0(B)$.

但是具体如何使用并不清楚.

关于此 L.E.S. 的理解:

① Deloop bundles $F \to E \to B$ to fiber sequence

$F \to E \to B \to X$, where F can be viewed as ΩX.

Then $\pi_0(F) = \pi_0(\Omega X) = [S^0, \Omega X] = [\Sigma S^0, X] = [S^1, X] = \pi_1(X)$ is a group. So $\pi_1(B) \to \pi_0(F) = \pi_1(X)$ is a group homo where exactness is well-defined. In fact, the long exact sequence 是这样得到的:

we have a bundle $F \to E \to B$

$\rightsquigarrow$

we have fiber sequence

$\cdots \to \Omega^2 B \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B$

$\rightsquigarrow$ apply π_0 $\quad$ ($\pi_0(\Omega^n X) = [S^0, \Omega^n X] = [\Sigma^n S^0, X] = [S^n, X] = \pi_n(X)$)

$\cdots \to \pi_n(B) \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to \pi_0(F) \to \pi_0(E) \to \pi_0(B)$.

关于 fiberation, Σ, Ω 可见这篇: https://people.math.binghamton.edu/malkiewich/fibration_sequences.pdf

② 关于这个 L.E.S. 的一个疑问: $\pi_0(F)$, $\pi_0(E)$, $\pi_0(B)$ may not be groups, 这时的 exact 如何理解? https://math.stackexchange.com/questions/2477364/interpreting-a-long-exact-sequence-of-homotopy-

当 F, B, E 是 pointed sets 时 $\pi_0(F)$, $\pi_0(B)$, $\pi_0(E)$ 也是 pointed set. pointed set 的 kernal 有定义, 因此 exactness 也有定义.

Question: ① 细读 Hatcher 参考.

② $\pi_n(\mathbb{R}^m) = ?$

▲ 一个可行的办法: 只要证 $E \not\cong B \times \mathbb{R}^n$ 就一定不 trivial.

可以证 $\pi_n(E) \not\cong \pi_n(B \times \mathbb{R}^n)$ or 证 $H_n(E) \not\cong H_n(B \times \mathbb{R}^n)$

关于 check $\pi_n(E) \not\cong \pi_n(B \times \mathbb{R}^n)$ 可以通过 π_1 作用在 π_n 上的 standard action 的结构判断. 关于 π_1 作用在 π_n 上的理解见: https://mathoverflow.net/questions/19775/different-way-to-view-action-of-fundamental-group-on-higher-homotopy-groups?_gl=1*q0156g*_ga*MTg0ODczNzkzNS4xNzIwNjgxMzM2*_ga_S812YQPLT2*MTcyMTIwNTQyNS4xMy4xLjE3MjEyMDYwNzMuMC4wLjA.

① $\tilde{X}$ is a covering of X, then $\pi_n(X) = \pi_n(\tilde{X})$, for $n \geq 2$.

于是 $\pi_1(X)$ acting on $\pi_n(X)$ can be viewed as $\pi_1(X)$ acting on $\pi_n(\tilde{X})$, which is a covering transformation

Given $p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$, we have:

an iso $\tilde{p}: \mathrm{Deck}(\tilde{X}) \xrightarrow{\sim} \pi_1(X, x_0)$, see https://math.stackexchange.com/questions/1446259/deck-transformations-of-universal-cover-are-isomorphic-to-the-fundamental-group

and a mor $p_*: \pi_n(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_n(X, x_0)$.

For $\alpha \in \pi_1(X, x_0)$, $\rho \in \pi_n(X, x_0)$, we have

$$\begin{cases} \tilde{\alpha} \text{ s.t. } \tilde{p}(\tilde{\alpha}) = \alpha & \Rightarrow \tilde{\alpha}_* : \pi_n(\tilde{X}, x_0) \longrightarrow \pi_n(\tilde{X}, x_0). \\ \tilde{\rho} \text{ s.t. } p_* \tilde{\rho} = \rho & \end{cases}$$

So $\alpha \cdot \rho = p_* \tilde{\alpha}_* (\tilde{\rho})$

③ If G is a topo grp, then $\pi_0(G)$ can act on $\pi_n(G)$ by conjugation. This structure may also be useful to see whether two spaces are homotopic.

- holonomy group might be a tool for judging.

① Let (M, g) be a Riemann manifold where g is the metric. Let L be a loop in M based at p. There is a linear transformation of T_pM associated to the loop: map $v \in T_pM$ to $v' \in T_pM$, where v' is the parallel transformation along the loop of v.

The holonomy group associated to p is a subgroup of $GL(T_pM)$ consisting of linear transformations associated to a loop based at p.

(这是参考 https://mathworld.wolfram.com/HolonomyGroup.html 给出的我理解的定义, it may be wrong :). Moreover, the gif in this website is very cool and very similar to the case of evolution of eigenvectors. That's why I think holonomy could be useful. Drawback: Unfamiliar tools of differential geometry :( )

② parallel transformation preserves Riemann metric, so the holonomy group $Hol(M) \subseteq O(n)$.

On an orientable manifold, $Hol(M) \subseteq SO(n)$.

* Manifold orientation https://mathworld.wolfram.com/ManifoldOrientation.html

In Hatcher we've known oriented in algebraic version. But the following geometric version is more enlighting. An orientation on an n-dim manifold is given by a nowhere vanishing differential n-form. Alternatively, it's an bundle orientation for the tangent bundle. (for bundle orientation, see https://mathworld.wolfram.com/BundleOrientation.html)

Some useful facts:

1. M is a codimentional one submanifold of $\mathbb{R}^n$.
   M is orientable iff it has a unit normal vector field.
2. Complex manifolds are always orientable.

(To do: Collect good properties of complex manifold)

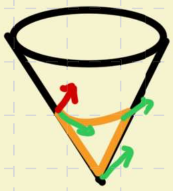
3. Any unoriented mf. has a double cover which is oriented

(To do: Collect properties covering spaces can "repair")

③ On a flat manifold, homotopic loops give same linear transformation —— $(T_pM, \rho: \pi_1(M,p) \to GL(T_pM))$
$L \longmapsto$ linear tranf. associated to L
is a representation of $\pi_1(M,p)$!

④ What does holonomy measure? https://math.stackexchange.com/questions/4175314/what-does-holonomy-measure

Let's start from a misunderstanding. One may think that flat manifold can only have trivial holonomy. It's not true and the contraexample is

the cone  (可能理解有误. Ref中的cone要挖掉 vertex 且 Loop 是 $z=c$)

holonomy measures "how many ways" there exists to go from a point and come back to it. The way to measure differences between two ways is closely related by curvature. holonomy combines the information of curvature and topology. Actually, the non-trivial holonomy of a flat m.f. contains the topo of this m.f.

* Further discussion:

1. There is a surjection $\pi_1(M) \to \mathrm{Hol}(g)/\mathrm{Hol}^0(g)$ where $\mathrm{Hol}^0(g)$ is the (full) holonomy grp and $\mathrm{Hol}^0(g)$ is the reduced holo. grp. Hence holo. grp encodes $\pi_1(M)$.

2. Roughly speaking, the smaller the holo. grp, the flatter the m.f.

3. holonomy principal: The holonomy grp completely detemines the existence (or non-existence) of <u>parallel vector fields</u>, <u>parallel differential forms</u>, <u>parallel spinor fields</u>.

4. Holo. grp. of g can be used to judge whether there is a decomposition $g = g_1 \times g_2$ for some Riemannian metric g_1, g_2.

5. Judge whether your m.f. is a kahler m.f.
   (Extra compatible geometric structure)

以下若干点的Ref: https://math.stackexchange.com/questions/3111658/what-needs-to-be-done-to-prove-that-a-vector-bundle-is-trivial

Let E be a vector bundle of rank n over a sufficiently nice, connected space.

[Def]: E is trivial iff $E \cong X \times \mathbb{R}^n$ where $\pi: E \to X$ is the bundle.

* bundle是否trivial与所处的category有关.
可见 fiberwise homotopy cat:
https://link.springer.com/content/pdf/ (意义不大,有空再看)

▲ **Independent Sections:** E is trivial iff it admits n sections which are linearly independent at every point.

Proof sketch: A trivialization defines n independent sections via the images of $X \times \{e_i\}$. Independent sections $\{\sigma_1, \ldots, \sigma_n\}$ define a basis for each fibre, inducing isomorphisms $E_x \cong \mathbb{R}^n$ which combine into a trivialization.

Corollary (Real line bundles): A one-dimensional real vector bundle is trivial iff it is orientable.

Proof: A non-zero section of a one-dimensional real vector bundle induces an orientation and vice-versa.

→ 把证 E trivial 化归到一个 vector bundle 最多有几个 independent sections 的问题, the so called "Vector field problem"

它在 special case 下的研究见

https://projecteuclid.org/journals/acta-mathematica/volume-128/issue-none/Vector-fields-with-finite-singularities/10.1007/BF02392157.full

▲ Tangent bundle of a Lie grp is trivial. (微分流形的一个结论)

▲ 尝试证 $X \to BG$ is null-homotopy 来证明是 X 上的 trivial bundle ($[X, BG] \cong \mathrm{Prin}_G X$)

1. 判断一个 map 是否 null homotopy —— obstruction theory. 但需 obstruction grps all vanish, it doesn't work in our case!

2. For vector bundle, the structure grp is $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$. X is paracompact, so we can choose a metric and reduce structure grp to $O(n)$ or $U(n)$.
于是参 $X \to BG$ null-homotopy 化归为判断 $X \to BO(n)$ (or $BU(n)$) 是否 null-homotopy.
关于 structure grp 和 reduction 见这篇 4页小 note:

http://faculty.bicmr.pku.edu.cn/~guochuanthiang/QT24/Week10.pdf

▲ X contractible, then any vector bundle over X is contractible.

▲ Real line bundles are classified by Stiefel-whitney class $H^1(X;\mathbb{Z}_2)$. https://www.math.umd.edu/~daylaian/sw_classes.pdf

Complex line bundles are classified by first Chern class $H^2(B;\mathbb{Z})$. https://web.ma.utexas.edu/users/gs29722/Chern1.pdf

* 只有line bundle才可用上述办法区分. 对于低rank bundle的分类研究见 https://dml.cz/bitstream/handle/10338.dmlcz/128427/CzechMathJ_43-1993-4_14.pdf

▲ **Proposition 4.1:** Let $E \longrightarrow X$ be a differentiable vector bundle. Then there is a finite open covering $\{U_\alpha\}, \alpha = 1, \ldots, N$, of X such that $E|_{U_\alpha}$ is trivial.

▲ https://math.stackexchange.com/questions/4409482/examples-of-spaces-with-only-trivial-vector-bundles (大片看不懂抄个结论先)

Proposition: Suppose B is a closed manifold for which every vector bundle over B is trivial. Then each of the following must be true:

1. B must be orientable.
2. B must be odd dimensional.
3. The first integral homology group must vanish.
4. B must have the rational homology of a sphere.