- $V(S_1) \cup V(S_2) = V(S_1 \cdot S_2)$

  $\bigcap_i V(S_i) = V(\bigcup_i S_i)$

  闭集有限并还是闭集.(一个理解是无穷乘积没定义)

- Any affine var can be written as zero locus of finite many polys.

  (That's because: A is Notherian $\Leftrightarrow$ every ideal I of A is finitely generated
  多项式环 $k[x_1,\dots,x_n]$ 是 Noetherian ring,
  其商环 $A(X) = k[x_1,\dots,x_n]/I(X)$ 也是 Noetherian.)

  $$V(S) = V(\langle S\rangle) = V(\{p_1f_1 + p_2f_2 + \dots + p_kf_k \mid f_i \in S_i,\ p_i \in k[x_1,\dots,x_n]\})$$

  $\langle S\rangle \trianglelefteq k[x_1,\dots,x_n]$ Noetherian, so $\langle S\rangle$ can be finitely generated.

  $$= V(\langle f_1,\dots,f_k\rangle)$$

- $V(\sqrt{J}) = V(J)$

  Exp: $J = \langle (x-a_1)^{k_1}(x-a_2)^{k_2}\cdots(x-a_r)^{k_r}\rangle$

  $$V(J) = V(\sqrt{J}) = V(\langle (x-a_1)\cdots(x-a_r)\rangle)$$

  Ideals of all polys vanishing at $a_1,\dots,a_r$ (with any order)

  $\Downarrow$

  affine var. 不记住重数信息,这是其与 scheme 的重大区别.

- $V(J_1) \cup V(J_2) = V(J_1J_2) \overset{*}{=} V(J_1 \cap J_2)$

  $V(J_1) \cap V(J_2) = V(\underbrace{J_1 \cup J_2}_{\text{as a set}}) = V(\langle J_1 \cup J_2\rangle) = V(J_1 + J_2)$

  $\Uparrow$

  理想的运算 $\langle J_1 \cup J_2\rangle = J_1 + J_2$.

  More dual then set-version
  $V(\cdot) \cup V(\cdot) = V(\cdot \cap \cdot)$
  $V(\cdot) \cap V(\cdot) = V(\underbrace{\cdot + \cdot}_{\langle \cup \rangle})$

  pf for $*$:

  "$\supseteq$" $J_1J_2 \subseteq J_1 \cap J_2$ so $V(J_1J_2) \supseteq V(J_1 \cap J_2)$

  "$\subseteq$" It suffices to show $V(J_1 \cap J_2) \supseteq V(\sqrt{J_1J_2}) = V(J_1J_2)$.

  $\forall f \in J_1 \cap J_2$, we have $f^2 \in J_1J_2$. so $f \in \sqrt{J_1J_2}$. Hence $J_1 \cap J_2 \subseteq \sqrt{J_1J_2}$.
  Then $V(J_1 \cap J_2) \supseteq V(\sqrt{J_1J_2})$

- $VI(X) = X$, $IV(J) = \sqrt{J}$.

$$\{\text{affine var in } \mathbb{A}^n\} \xleftrightarrow{1:1} \{\text{radical ideals in } k[x_1, \cdots, x_n]\}$$

$$X \longmapsto I(X)$$

$$V(J) \longmapsfrom J$$

Rmk: 要求 alg closed. 原因是 $\sqrt{J} \supseteq IV(J)$ 只有 k alg closed 才满足.
k 不 closed 定理失效的例子:

$J = x^2+1 \subseteq \mathbb{R}[x]$ $\quad V(J) = \phi$. $\quad IV(J) = \mathbb{R}[x]$. $\quad IV(J) \supsetneq \sqrt{J}$.

Rmk: proof $VIX \subset X$ is interesting.

$X = V(J)$ for some $J \trianglelefteq k[x_1, \cdots, x_n]$.

w.t.s. $VIV(J) \subset V(J) = V(\sqrt{J})$ $\rightsquigarrow$ w.t.s. $\underset{\text{easy}}{\sqrt{J} \subset IV(J)}$

- $k[x]$ is PID. so $J \subseteq k[x]$ can only be $J = \langle f \rangle \underset{\text{alg closed.}}{=} \langle (x-a_1)^{k_1} \cdots (x-a_r)^{k_r} \rangle$

So $V(J) = V(\langle (x-a_1)\cdots(x-a_r) \rangle)$
$= \bigcup_{i=1}^{r} V(\langle x-a_i \rangle)$ $\rightsquigarrow$ Closed set in $\mathbb{A}^1$ is a union of finite points.

- $\{\text{points in } \mathbb{A}^n\} \xleftrightarrow{1:1} \{\text{maximal ideals in } k[x_1, \cdots, x_n]\}$.

It's a restriction of the bijection in Hilbert Null thm.

(△ a pt is minimal non-empty varieties in $\mathbb{A}^n$. so $I(a)$ is maximal $\subseteq k[x_1, \cdots, x_n]$

△ maximal ideal is radical $\underset{\text{maximal}}{J} \subseteq \sqrt{J} \Rightarrow J = \sqrt{J}$

$\Downarrow$

△ $\{\text{maximal ideals}\} \subseteq \{\text{radical ideals}\}$ 所以可以把 bijection 限制到 $\{\text{pts in } \mathbb{A}^n\}$ 上.)

- $I(X_1 \cup X_2) \overset{\text{trivial}}{=} I(X_1) \cap I(X_2)$

$I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$

$$\begin{aligned} I(X_1 \cap X_2) &= I(VI(X_1) \cap VI(X_2)) \\ &= I(V(I(X_1) + I(X_2))) \\ &= \sqrt{I(X_1) + I(X_2)} \end{aligned}$$

Rmk1: Obtained by "1 to 1 correspondence"

prop of $V(\cdot)$ $\rightsquigarrow$ prop of $I(\cdot)$

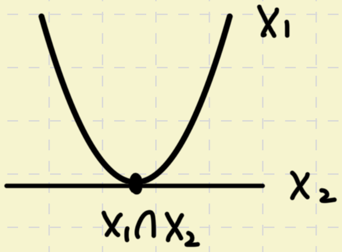
$V_p(\cdot)$ $\quad I_p(\cdot)$

$P(\cdot)$ $\quad C(\cdot)$

Rmk2: Geometric picture $X_1, X_2 \subseteq \mathbb{A}^2_{\mathbb{C}}$ with ideals

$I(X_1) = \langle x_2 - x_1^2 \rangle$ and $I(X_2) = \langle x_2 \rangle$

绘制实部如下:



△ $I(X_1 \cap X_2) = I(0) = \langle x_1, x_2 \rangle$

($I(a) = \langle x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n \rangle$)

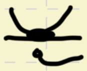
△ $I(X_1) + I(X_2) = \langle I(X_1) \cup I(X_2) \rangle$

$= \langle x_2 - x_1^2, x_2 \rangle$

$= \langle x_1^2, x_2 \rangle$ it's not radical.

△ "Strech" — why $I(X_1) + I(X_2)$ is non-radical?

X_1 与 X_2 在线性近似下是 x_1轴. 所以我们可以想象把交点沿x_1轴strech无穷小量.



新的 $x_1 \cap x_2$.

对于新 $x_1 \cap x_2$, x_1 不 vanish, 只有 x_1^2 才 vanish. 所以 $I(X_1) + I(X_2)$ 记录了交点附近的信息 (tangential 信息)

但 $\sqrt{I(X_1) + I(X_2)}$ 取 radical 忘记这些 tangential 信息.

- $\sqrt{\langle 0 \rangle} = \langle 0 \rangle$ 意味着环没有幂零元.

  One way to compute $\sqrt{\langle 0 \rangle}$: $\underset{\sqrt{\langle 0 \rangle}}{\overset{\|}{I V(0)}} = I(\mathbb{A}^n)$

- 求坐标环的相容性 $X \subset Y \subset A^n$.

  用 $X \subset \mathbb{A}^n$ 求 "$A(X)$": $A(X) = k[x_1, \ldots, x_n] / I(X)$

  用 $X \subset Y$ 求 "$A_Y(X)$": $A_Y(X) = A(Y) / I_Y(X)$

  二者相同, i.e., $A(X) \cong A_Y(X)$.

  $A(X)$ 指 X 上所有 poly funcs, 这是一个内蕴的概念, 无论把 X 套在哪个 aff. var. 中都求出一样的坐标环.

- 尽管称 $A(X)$ 为 coordinate ring, 我们总将之视作 k-alg.

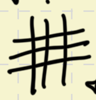
  $A(X) = k[x_1, \ldots, x_n] / I(X)$.

  $A(X)$ 中的元素称为 poly. fun.

  Rmk: K closed. A poly. fun. in $x_1, \ldots, x_n$ is determined uniquely by its values at pts of var.

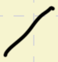
- Zariski 拓扑 ≠ 乘积拓扑.

  $\mathbb{A}^1 \times \mathbb{A}^1$ 若只考虑乘积拓扑



形如左图, 有限条横线竖线相交.

但 $\mathbb{A}^1 \times \mathbb{A}^1$ 中的闭集还包括



对角线 $V(x_2 - x_1)$

- {zariski closed subset} $\subseteq$ {classical closed set}

e.g. $\mathbb{A}^1_{\mathbb{C}} = \{\{\text{finite pts}\}$ is closed sets.

$A = \{x : \mathbb{A}^1_{\mathbb{C}} : |x| \leq 1\}$ not closed in $\mathbb{A}^1_{\mathbb{C}}$ and $\overline{A} = \mathbb{A}^1_{\mathbb{C}}$

Closed subsets in Zariski topo is very small (finite points)
open subsets in Zariski topo is very big (以至于 $\mathbb{A}^1 \times \mathbb{A}^1$ 不 Hausdorff)

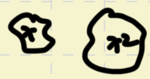
- 为什么 aff var 间的 mor 不能只考虑连续映射，而考虑 mor of ringed space?

  $f: \mathbb{A}^1 \to \mathbb{A}^1$ 的 map 可以非常 weired，但只要 f 是单射，zariski topo 会判定它连续.
  由于 zariski topo 与 classical topo 非常不同，导致很多经典拓扑定义的概念在 zariski topo 中变得 useless. zariski topo 下有新的、与 classical topo 对应的.更有用的概念.

- △ disconnected $V(x_1) \cup V(x_1 - 1)$ $||$ $\overset{\text{nontrivial}}{X = X_1 \cup X_2}$, $X_1 \cap X_2 = \phi$
  $\Downarrow$
  reducible $V(xy)$ $+$ $\overset{\text{nontrivial}}{X = X_1 \cup X_2}$

△ irr $\Rightarrow$ connected.

- △ X is disconnected aff. var. with $X = X_1 \cup X_2$, $\overset{\text{disjoint closed}}{X_1, X_2 \subsetneq X}$. Then $A(X) \simeq A(X_1) \times A(X_2)$



pf1: $f \mapsto (f|_{X_1}, f|_{X_2})$

pf2: $I(X_1) + I(X_2) = 1$

$(I_i + R_j = R, \forall i \neq j,\ R/\bigcap_i I_i \simeq \prod_i R/I_i) \rightsquigarrow A(X)/I(X_1 \cup X_2) \cong A(X)/I(X_1)+I(X_2) \overset{中国剩余定理}{\cong} \underbrace{A(X)/I(X_1)}_{A_X(X_1) = A(X_1)} \times \underbrace{A(X)/I(X_2)}_{A_X(X_2) = A(X_2)}$

△ $\phi \neq X$ is irr iff $\begin{cases} A(X) \text{ is an integral domain} \\ I(X) \text{ is prime ideal} \end{cases}$

$*$ $f_1 f_2 = 0 \rightsquigarrow X = V(0) = V(f_1 f_2) = V(f_1) \cup V(f_2)$ red. 因式分解对应 var 的分解.

△ Since prime ideal is radical, so we have restriction to Hilbert null thm

{non-empty irr aff var of Y} $\overset{1:1}{\longleftrightarrow}$ {prime ideals in A(Y)}

- Aff. var. X given by linear equs is irr In particular $\mathbb{A}^n = V(0)$ is irr.

  $X = V(a_1 x_1 + \cdots + a_n x_n)$. so $A(X) = k[x_1, \cdots, x_n] / \langle a_1 x_1 + \cdots + a_n x_n \rangle$
  $\overset{\text{say } a_1 \neq 0}{\simeq} k[x_2, \cdots, x_n]$ is integral domain.

(多条 linear equs 一样，总可解出若干变量[无解就是 $k[x_1, \cdots, x_n]$])

- $X \subsetneq Y$ 则 $I_Y(X) \neq 0$. (由 bijection, 若是 0 则 $X = V_Y I_Y X = V_Y(0) = Y$.)
  大空间.

- Noetherian space: 没有闭子集无穷降链.

infinite decreasing chain of closed sets $X_0 \supsetneq X_1 \supsetneq \cdots$ $\leftrightarrow$ infinite increasing chain $I(X_0) \subsetneq I(X_1) \subsetneq \cdots$ in $A(X)$. Noetherian. ← Impossible.

(USE DUALITY to prove.)

所有 aff. var. 是 Noetherian space.

- subspace of Noetherian space is Noetherian.

$A \cap X_0 \supsetneq A \cap X_1 \supsetneq A \cap X_2 \supsetneq \cdots$ ← chain of closed sets in A.

$\not\Downarrow$

$X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$

$X_0 \supsetneq X_0 \cap X_1 \supsetneq X_0 \cap X_1 \cap X_2 \supsetneq \cdots$

e.g.

X_1 与 X_0 间可以没有包含关系

$$\left(\begin{array}{l} A \cap X_1 \supsetneq A \cap X_2 \\ A \cap X_1 \cap X_1 \supsetneq A \cap X_2 \cap X_1 \\ A \cap X_1 \supsetneq A \cap (X_2 \cap X_1) \\ X_1 \supsetneq X_2 \cap X_1 \quad \text{(有消去律)} \end{array}\right)$$

- Every Noetherian topo space can be written as a finite union $X = X_1 \cup \cdots \cup X_r$ of non-empty irr closed subsets.
  Assume $X_i \not\subset X_j$ for all $i \neq j$, $X_1, \cdots, X_r$ are unique (up to permutation)

(Existence: 如果可以一直拆分, 则会得无穷降链与 Noetherian 条件矛盾.

Uniqueness: $X = X_1 \cup \cdots \cup X_r = X_1' \cup \cdots \cup X_s'$.

$X_i \subset \bigcup_j X_j'$ so $X_i = \bigcup_j (X_j' \cap X_i)$. 由 X_i irr, $X_i = X_j' \cap X_i$, i.e., $X_i \subset X_j'$ for some j. In the other way $X_j' \subset X_k$ for some k.

$X_i \subset X_j' \subset X_k \xRightarrow{i \neq k \text{时} X_i \not\subset X_k} X_i = X_k = X_j'$.)

-

$\{$non-empty irr aff var of $Y\} \xleftrightarrow{1:1} \{$prime ideals in $A(Y)\}$

$\}$ Restriction

$\{$Maximal non-empty irr aff var of $Y\} \xleftrightarrow{1:1} \{$Minimal prime ideals in $A(Y)\}$

$\quad \| $

irr components of Y

- $U_1, U_2, U \overset{\text{open}}{\subseteq} \underset{\text{irr}}{X}$ Then $U_1 \cap U_2 \neq \phi$ $(U_1^c \cup U_2^c \neq X)$, $\overline{U} = X$ (A irr $\Leftrightarrow \bar{A}$ irr, so U irr)

Hausdorff nonsense in Zariski topo

$\left(\forall \underset{\text{closed}}{Y} \supseteq U, \underset{\text{irr}}{X} = Y \cup (X \setminus U) \text{ is trivial}\right)$

$X \setminus U \neq X$ so $Y = X$

Zariski: open sets are big!

- △ The dimension $\dim X \in \mathbb{N} \cup \{\infty\}$ is the supremum over all $n \in \mathbb{N}$ s.t. there is a chain $\emptyset \neq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \subset X$ of length n of irr closed subsets $Y_1, \dots, Y_n$ of X

※ Idea of this def: if X irr. Any closed subset of X not equal to X should have smaller dimension.

△ $Y \subseteq X$ (irr closed) $\operatorname{codim}_X Y$ is the supremum over all n s.t. there is a chain $Y \subset Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \subset X$ of irr closed subsets

△ dim for aff var, codim only defined for irr closed sets of an aff var (irr sub aff var)

* '$\subset X$', because X may not irr.

△ $\dim X = \dim A(X)$, $\operatorname{codim}_Y X = \operatorname{codim}_{A(X)} I(Y)$,

In particular, since $A(X)$ is finite generated k-alg, dim of aff var and codim of irr aff var are always finite

△ $A \subseteq X$ (arbitrary set) Then $\dim A \le \dim X$

- $\emptyset \neq X, Y$ irr aff. var.
  ① $\dim X \times Y = \dim X + \dim Y$ $(\dim \mathbb{A}^n = n \dim \mathbb{A}^1 = n)$
  ② If $Y \subseteq X$, we have $\dim X = \dim Y + \operatorname{codim}_X Y$ $(\operatorname{codim}_X \{a\} = \dim X, \forall a \in X)$
  ③ $0 \neq f \in A(X)$, every irr component of $V(f)$ has codim 1 in X.

What if X is not irr? $X = X_1 \cup X_2 \cup \cdots \cup X_r$.

① $\dim X = \max\{\dim X_1, \dots, \dim X_r\}$

idea: show $\begin{cases} \dim X \geq n \Rightarrow \max_i\{\dim X_i\} \geq n \\ \max_i\{\dim X_i\} \geq n \Rightarrow \dim X \geq n. \end{cases}$

利用此idea 外加 irr 易证.

REASON: $\dim X \geq n \Rightarrow \max_i\{\dim X_i\} \geq n$

右手取sup $\Rightarrow$ $\underset{\dim X}{\sup n} \leq \max_i\{\dim X_i\}$ $\dim X \geq n$.

sup是最小上界

so $\dim X \leq \max_i\{\dim X_i\}$

互为对方的MAX

② $\dim X = \sup\{\operatorname{codim}_X\{a\} : a \in X\}$ <u>dim X is sup local dim !</u>

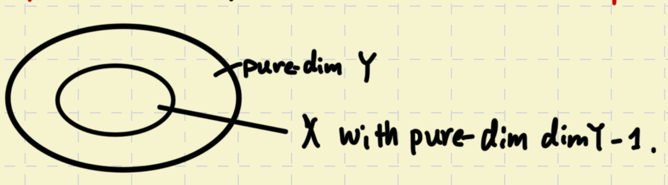
idea: show $\dim X \geq n \Rightarrow \sup\{\operatorname{codim}_X\{a\}\} \geq n \to \sup(\operatorname{codim}_X\{a\}) \geq \sup n = \dim X$

$\sup \operatorname{Codim}_X\{a\} \geq n \Rightarrow \dim X \geq n \to \dim X \geq \sup n = \sup \operatorname{codim}_X\{a\}$

③ $\operatorname{codim}_X\{a\} = \max\{\dim X_i : a \in X_i\}$ (经过a的不可约分支最大维数)

- △ A Noetherian topo space X is said to be pure dim n if every irr component of X has dim n.

△ Hypersurface X is an aff var in a pure-dim aff. Y with pure-dim $\dim Y - 1$



△ Thm: R is a Noetherian integral domain (e.g. $A(X)$ of an irr aff. var. X).
TFAE (a) Every prime ideal of codim 1 in R is principal
(b) R is a unique factorization domain (可以写成 product of prime elements)

△ Ideal of hypersurface.

* $f\neq 0$, $V(f)\subseteq \underset{\sim}{X}_{irr}$ then every irr component of $V(f)$ has codim 1, i.e., $V(f)$ is a hypersurface with pure dim $\dim X - 1$.

* hypersurface X in $\mathbb{A}^n$

① X irr.

$I(X) \lhd k[x_1,\cdots,x_n]$ prime
codim $I(X)$ = codim X = 1
$k[x_1,\cdots,x_n]$ is U.F.D.

proceding thm → $I(X)$ is principal, i.e., $I(X)=\langle f\rangle$. So $X=V(f)$.

② X red.

$X = X_1\cup X_2\cup\cdots\cup X_r$
$I(X_i) \trianglelefteq k[x_1,\cdots,x_n]$ prime
codim $I(X_i)$ = codim X_i = 1
$k[x_1,\cdots,x_n]$ is U.F.D.

→ $I(X_i)=\langle f_i\rangle \rightsquigarrow X_i = V(f_i)$

$X=\bigcup V(f_i) = V(f_1\cdots f_r) =: V(f)$

Hypersurface in $\mathbb{A}^n$ (whether irr. or red.) can be written as a zero locus of a single poly f. And f is unique up to units (U.F.D.)
Degree of f is called the degree of X.

- Def of regular functions: $U\underset{open}{\subseteq} X$ $\varphi: U\to K$, $\forall a\in U$ $\exists f,g\in A(X)$ $a\in U_a\subseteq U$ s.t. $\varphi(x)=\frac{g(x)}{f(x)}$, $f(x)\neq 0$, $\forall x\in U_a$. The set of all such regular functions on U will be denoted $\mathcal{O}_X(U)$.

△ we write "$\varphi=\frac{g}{f}$ on U_a" for simplicity. Note that $\frac{g}{f}$ is point wise quotient of functions, NOT an element in a localized ring.

△ By def we know 不同U_a相交处表达式相同(因为都是φ,只是不同写法)

△ Why we need so complex def?
What if we def regular functions as quotient of functions, NOT Locally quotient of functions?
It's NOT useful. 一个$\varphi: U\to K$ 可能没有全局的 quotient of funs 表达式,但有 locally quotient of functions 的表达式.

e.g. $X=V(x_1x_4 - x_2x_3)\subset\mathbb{A}^4$, $U := X\setminus V(x_2,x_4)$

$\varphi: U\to K$, $(x_1,x_2,x_3,x_4)\longmapsto \begin{cases} x_1/x_2 & \text{if } x_2\neq 0 \\ x_3/x_4 & \text{if } x_4\neq 0\end{cases}$

φ没有 global quotient.

* Good point of view: Identify $\mathbb{A}^4 \cong M_{2\times 2}(K)$. Then $X=V(\det M)$,

i.e., $X=\{M\in M_{2\times 2}(k)\mid \operatorname{rank} M\leq 1\}$. (since $\det M=0$)

Then $U= X\setminus V(x_2,x_4)=\left\{M\in M_{2\times 2}(k)\ \middle|\ \begin{array}{l}\text{second column vector } \begin{pmatrix}x_2\\x_4\end{pmatrix}\neq\begin{pmatrix}0\\0\end{pmatrix}\\ \text{first column vector } \begin{pmatrix}x_1\\x_2\end{pmatrix}=\lambda\begin{pmatrix}x_2\\x_4\end{pmatrix}\end{array}\right\}$

Then $\varphi: U\to k$ $\quad \varphi\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix}=\lambda$.

- $\underset{\text{open}}{U\subseteq} X$, $\varphi\in\mathcal{O}_X(U)$, $V(\varphi)$ is closed in U.

($\varphi\notin A(X)$, it's not obvious $V(\varphi)$ is closed.)

Idea: Decompose to small parts.

pf: $\forall a\in U$ $\exists a\in U_a\subseteq U$ s.t. $\varphi=\dfrac{g_a}{f_a}$ on U_a.

$U_a\setminus V(\varphi)=\{x\in U_a\mid \varphi(x)\neq 0\}=\{x\in U_a\mid g_a(x)\neq 0\}=U_a\setminus V_a$ open.

So $U\setminus V(\varphi)=\bigcup_a\left(U_a\setminus V(\varphi)\right)$ is open.

- $\varphi\in\mathcal{O}_X(U)$ is conti.

Pf: $\varphi: U\to k=\mathbb{A}^1_k$. w.t.s. $\varphi^{-1}(\{\text{finite pts}\})=\bigcup_{pt}\varphi^{-1}(pt)$ is Closed. It suffices to show $\varphi^{-1}(c)$ closed for any $c\in k$. It will reduce to show $\varphi^{-1}(0)$ closed, since let $\tilde\varphi=\varphi-c$, then $\tilde\varphi^{-1}(0)=\varphi^{-1}(c)$.

Idea: Decompose to small parts.

$\forall a\in U$, $\exists a\in U_a\subseteq U$ s.t. $\varphi=\dfrac{g_a}{f_a}$ on U_a. $U_a\setminus\varphi^{-1}(0)=\left\{x\in U_a\ \middle|\ \dfrac{g_a(x)}{f_a(x)}\neq 0\right\}=\{x\in U_a\mid g_a(x)\neq 0\}=\underset{\text{open in } U}{D(g_a)\cap U_a}$

so $U\setminus\varphi^{-1}(0)=\bigcup_a U_a\setminus\varphi^{-1}(0)$ is open and thus $\varphi^{-1}(0)$ closed.

- ~~例~~ $\underbrace{U\subset}_{\text{open}} V\subset \underset{\text{irr}}{X}$ $\quad\varphi_1,\varphi_2\in\mathcal{O}_X(V)$. If $\varphi_1|_U=\varphi_2|_U$, then $\varphi_1=\varphi_2$.

$(\varphi_1-\varphi_2)|_U=0\Rightarrow U\subseteq\underbrace{(\varphi_1-\varphi_2)^{-1}(0)}_{\text{closed by conti}}\Rightarrow\underbrace{\bar U}_{\text{closure of } U \text{ in } V}\subseteq(\varphi_1-\varphi_2)^{-1}(0)\Rightarrow$ ($\bar V=X$ irr so V irr) V open in irr X, so V irr

$\Rightarrow U$ open in irr V, so $\bar U=V\Rightarrow V=\bar U\subseteq(\varphi_1-\varphi_2)^{-1}(0)\Rightarrow\varphi_1-\varphi_2|_V=0\Rightarrow\varphi_1=\varphi_2\in\mathcal{O}_X(V)$

GaGa principal: 解析几何与代数几何的结论可以共用.
在解析几何中有类似的Identity thm

- △ $D(f)\cap D(g)=D(fg)$ 开集有限交还是开集 无穷乘积没定义.

△ 任何aff. var. (closed set) 可写作有限多项式零点. $V(f_1,\cdots,f_k)$.

故任何开集 $U=X\setminus V(f_1,\cdots,f_k)=X\setminus\left(\cap V(f_i)\right)=X\cap\left(\cap V(f_i)\right)^c=\bigcup_i V(f_i)^c=\bigcup_i D(f_i)$

Distinguish open set: 拓扑基.

△ $\mathcal{O}_X(D(f))\cong A(X)_f=\left\{\dfrac{g}{f^n}\ \middle|\ g\in A(X),\ n\in\mathbb{N}\right\}$ * $D(f)$上的 regular fun 可以全局地写成多项式之商

set $f=1$, $\mathcal{O}_X(X)\cong A(X)$, X 上的 regular fun 是 poly, 不是 quotient of poly.

* NOT TRUE for k not alg closed. e.g., $\dfrac{1}{x^2+1}\notin A(\mathbb{A}^1_{\mathbb{R}})=k[x]$ BUT $\dfrac{1}{x^2+1}\in\mathcal{O}_{\mathbb{A}^1_{\mathbb{R}}}(\mathbb{A}^1_{\mathbb{R}})$ contradicting to $\mathcal{O}_X(X)\cong A(X)$.

* 用于判断一个开集是否可以写成单个多项式的非零点.

在之前例子中, $U = V(x_1x_4 - x_2x_3) \setminus V(x_2, x_4)$. 若 $U = D(f)$, 则其上 regular function $\varphi \in \mathcal{O}_X(U) \cong A(X)_f$ 有全局的多项式商表达式, 但事实上 φ 没有. 所以 U 必不形如 $D(f)$.

* Application: Every regular fun of $\mathbb{A}^2 \setminus \{(0,0)\}$ can be extended to a regular fun on $\mathbb{A}^2$.

Idea: Decompose to small parts. $\mathbb{A}^2 \setminus \{0,0\} = D(x_1) \cup D(x_2)$ ← $\mathcal{O}_X(D(f)) \cong A(X)_f$!

For any $\varphi \in \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \setminus \{(0,0)\})$, $\varphi|_{D(x_1)} \in \mathcal{O}_{\mathbb{A}^2}(D(x_1)) \cong k[x_1, x_2]_{x_1}$, say

$\varphi|_{D(x_1)} = \frac{f_1}{x_1^n}$, $f_1 \in k[x_1, x_2]$, $x_1 \nmid f_1$; Similarly, say $\varphi|_{D(x_2)} = \frac{f_2}{x_2^m}$, $x_2 \nmid f_2$.

On $D(x_1) \cap D(x_2)$, $\frac{f_1}{x_1^n} = \frac{f_2}{x_2^m}$ so $x_2^m f_1 - f_2 x_1^n = 0$ on $D(x_1) \cap D(x_2)$.

By Identity thm for regular functions, $x_2^m f_1 - f_2 x_1^n = 0$ on $\mathbb{A}^2$.

So $x_2^m f_1 = f_2 x_1^n$ on $\mathbb{A}^2$. If $m \neq 0$, we have $x_2 | f_2$; If $n \neq 0$, we have $x_1 | f_1$, (与假设矛盾)

so $m = n = 0$. Then $f_1 = f_2$ on $\mathbb{A}^2$, $\varphi = f_1 = f_2$ on $\mathbb{A}^2$

* Step 1 Show $\mathcal{O}_X(D(f)) = \{ \frac{g}{f^n} : g \in A(X), n \in \mathbb{N} \}$ (在集合层面等于 $A(X)_f$)

"$\supseteq$" obviously.

"$\subseteq$" $\forall \varphi \in \mathcal{O}_X(D(f))$, $\forall a \in D(f)$, $\exists\ a \in U_a \subseteq D(f)$, s.t. $\varphi = \frac{g_a}{f_a}$ on U_a for some $g_a, f_a \in A(X)$.

Rmk: We have open cover $D(f) = \bigcup_a U_a$. 但这样的 open cover 不好用, 我们希望得到形如 $D(h_a)$ 的 open cover.

U_a 可以用一堆 $D(h_a)$ cover, 不妨设 $U_a = D(h_a)$. 则 $\varphi|_{U_a} = \frac{h_a g_a}{h_a f_a} \begin{array}{c} \bar{g}_a := h_a g_a \\ \overline{} \\ \bar{f}_a := h_a f_a \end{array} \frac{\bar{g}_a}{\bar{f}_a}$

on $D(h_a) = U_a$. It's well defined because $f_a \neq 0$ on U_a, $h_a \neq 0$ on $D(h_a)$, so $\bar{f}_a \neq 0$ on $D(h_a)$. Besides $D(h_a) = D(\bar{f}_a)$, because $\bar{f}_a$ vanishes on $V(h_a)$ and not vanish on $D(h_a)$. As subspace of Noetherian space, $D(f)$ is Noetherian.

All Noetherian space is compact. So $D(f) = \bigcup D(h_a) = \bigcup D(\bar{f}_a)$ has finite cover

$D(f) = \bigcup_{i=1}^{s} D(\bar{f}_{a_i})$. 取补得 $V(f) = \bigcap_{i=1}^{s} V(\bar{f}_{a_i}) = V(\bar{f}_{a_1}, \cdots, \bar{f}_{a_s})$

取 I 得 $\underset{f \in}{\sqrt{\langle f \rangle}} = \sqrt{\langle \bar{f}_{a_1}, \bar{f}_{a_2}, \cdots, \bar{f}_{a_s} \rangle}$. so $f^n \in \langle \bar{f}_{a_1}, \cdots, \bar{f}_{a_s} \rangle$.

$\Rightarrow f^n = p_{a_1} \bar{f}_{a_1} + \cdots + p_{a_s} \bar{f}_{a_s}$

于是我们可以把 φ globally 写成多项式之商: $\forall x \in D(\bar{f}_{a_1}) \cap D(\bar{f}_{a_2}) \cdots \cap D(\bar{f}_{a_s})$

$$\varphi = \frac{\bar{g}_{a_1}}{\bar{f}_{a_1}} = \frac{\bar{g}_{a_2}}{\bar{f}_{a_2}} = \cdots = \frac{\bar{g}_{a_s}}{\bar{f}_{a_s}} = \frac{p_{a_1}\bar{g}_{a_1} + p_{a_2}\bar{g}_{a_2} + \cdots + p_{a_s}\bar{g}_{a_s}}{p_{a_1}\bar{f}_{a_1} + p_{a_2}\bar{f}_{a_2} + \cdots + p_{a_s}\bar{f}_{a_s}} =: \frac{g}{f^n}$$

$\frac{g}{f^n}$ 这个表达式可以延拓到其余部分, 因为 $V(\bar{f}_{a_j})$ 上 $\begin{cases} \bar{g}_{a_j} = 0 \\ \bar{f}_{a_j} = 0 \end{cases}$

e.g. $x \in D(\bar{f}_{a_1}) \cap D(\bar{f}_{a_2})$ 但在其子 $V(\bar{f}_{a_j})$ 里.

$$\frac{g}{f^n} = \frac{p_{a_1}\bar{g}_{a_1} + p_{a_2}\bar{g}_{a_2}}{p_{a_1}\bar{f}_{a_1} + p_{a_2}\bar{f}_{a_2}} = \frac{\bar{g}_{a_1}}{\bar{f}_{a_1}} = \frac{\bar{g}_{a_2}}{\bar{f}_{a_2}}$$

留下的都是等比例相加;
没留下的分子分母是 0.

*Step2 证明 $\mathcal{O}_X(D(f)) \cong A(X)_f$ as k-alg.

$$A(X)_f \longrightarrow \mathcal{O}_X(D(f))$$
$$\frac{g}{f^n} \longmapsto \frac{g}{f^n}$$

Formal fraction — Quotient of poly funs.

well-defined:

$\frac{g_1}{f^n} \sim \frac{g_2}{f^m}$ in $A(X)_f$, i.e.,

$\exists f^k$ s.t. $f^k(g_1 f^m - g_2 f^n) = 0$ in $A(X)$

$\Rightarrow$ on $D(f)$, $f \neq 0$, so $g_1 f^m - g_2 f^n = 0$

$\Rightarrow \frac{g_1}{f^n} = \frac{g_2}{f^m}$.

Surjective: Step 1.

Injective: $\frac{g}{f^n} = 0$ on $D(f) \Rightarrow g = 0$ on $D(f) \Rightarrow fg = 0$ on $X \Rightarrow f(g\cdot 1 - 0\cdot f^n) = 0$ in $A(X)$

$\left(X = \underbrace{V(f)}_{f=0} \cup \underbrace{D(f)}_{g=0}\right)$ $\Rightarrow \frac{g}{f^n} = \frac{0}{1}$

- presheaf $\mathcal{F}$ of rings consists of following data

(1) $\forall U \underset{\text{open}}{\subseteq} X$, $\mathcal{F}(U)$ is a ring

(2) $\forall$ inclusion $U \subseteq V$, a ring homo $\rho_{V,U} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$ called the restriction map satisfying

(1) $\mathcal{F}(\emptyset) = 0$

(2) $\rho_{U,U} = id$, $\forall U \overset{\text{open}}{\subseteq} X$

(3) Any inclusion $U \subset V \subset W$, $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$

*Elements in $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ over U

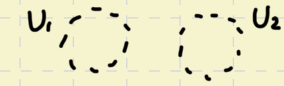
*(Sheaf) A presheaf $\mathcal{F}$ is called a sheaf if it satisfies glueing prop:

$U \underset{\text{open}}{\subseteq} X$, $U = \bigcup_{i \in I} U_i$ an open cover of U, $\varphi_i \in \mathcal{F}(U_i)$ fulfilling $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there is a unique $\varphi \in \mathcal{F}(U)$ s.t. $\varphi|_{U_i} = \varphi_i$ for $\forall i$.

* Local 性质构成 sheaf. $\mathcal{F}(U) = \{\varphi : U \to \mathbb{R}$ continous / locally const / differiable$\}$ 都是 sheaf.

Global 概念不构成 sheaf, 例如 const funs 粘不起来, 只是 presheaf.

$\mathcal{F}(U) = \{\varphi : U \to \mathbb{R}$ const$\}$ is not a sheaf. Consider $U = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$



$\varphi_1 : U_1 \to \mathbb{R}$, $x \mapsto 1$ $\qquad$ $\varphi_2 : U_2 \to \mathbb{R}$, $x \mapsto 0$

不存在 $U = U_1 \cup U_2$ 上的 const fun φ s.t. $\varphi|_{U_1} = \varphi_1$, $\varphi|_{U_2} = \varphi_2$

不满足 glueing prop.

- $U \overset{\text{open}}{\subseteq} X$, $\mathcal{F}$ is a presheaf on X. 则可构造 U 上的 presheaf $\mathcal{F}|_U$: $\mathcal{F}|_U(V) = \mathcal{F}(V)$

*When $U \underset{\text{open}}{\subseteq} X \underset{\text{open}}{\subseteq} Y$ ~ aff. var. $\qquad \mathcal{O}_X(U) = \mathcal{O}_Y(U)$ By restriction

- Sheaf = Collection of stalks

* Def: (stalks of (pre-)sheaves) $\mathcal{F}$ be a presheaf on X, $a \in X$
  the stalk of $\mathcal{F}$ is

$$\mathcal{F}_a := \{(U, \varphi) \mid U \subset X \text{ open with } a \in U, \varphi \in \mathcal{F}(U)\} / \sim$$

$(U, \varphi) \sim (U', \varphi') \Longleftrightarrow \exists$ open $a \in V \subset U \cap U'$ s.t. $\varphi|_V = \varphi'|_V$

* $\mathcal{F}_a$ inherits structure of k-alg of $\mathcal{F}(U)$.
* Elements of $\mathcal{F}_a$ are called germs of $\mathcal{F}$ at a. $(\mathcal{O}_X)_a =: \mathcal{O}_{X,a}$
* $\mathcal{F}_a$: "all functions on a."
* germs in stalks are something "local functions",包含local信息.
  e.g. sheaf of differentiable function over $\mathbb{R}$, a germ at a pt $a \in \mathbb{R}$ allows to compute all the derivatives at a.

* 数学描述"local function"

$$\mathcal{O}_{X,a} \cong A(X)_{I(a)} = \{\frac{g}{f} : f, g \in A(X), \underbrace{f(a) \neq 0}_{\Leftrightarrow f \notin I(a)}\}.$$

As a local ring, $\mathcal{O}_{X,a}$ has unique maximal ideal.

Rmk: ($\mathcal{O}_{X,a}$, a 上 fun 以 $f(a)\neq 0$ 的 f 作分母; 同样 $\mathcal{O}_X(D(f))$ 以 f 作分母)

Pf: (Routine check)

$$\rho: A(X)_{I(a)} \longrightarrow \mathcal{O}_{X,a}$$

$$\frac{g}{f} \longmapsto \overline{(\underbrace{D(f)}_{\text{含a开集}}, \underbrace{\frac{g}{f}}_{\text{in } \mathcal{O}_X(D(f))})} \leftarrow \text{模掉等价关系}$$

① well defined

Trick: 放到D(h)上让式子变 $g_1f_2 - f_1g_2 = 0$

$\frac{g_1}{f_1} \sim \frac{g_2}{f_2} \Rightarrow \exists \underbrace{h \notin I(a)}_{\Leftrightarrow a \in D(h)}$ s.t. $h(g_1f_2 - f_1g_2) = 0$ in $A(X)$

In $\underbrace{D(h) \cap D(f_1) \cap D(f_2)}_{\text{smaller n.b.h.}} \subseteq D(f_1) \cap D(f_2)$, $h \neq 0$ so $g_1f_2 - f_1g_2 = 0$, i.e., $\frac{g_1}{f_1} = \frac{g_2}{f_2}$ on $D(h) \cap D(f_1) \cap D(f_2)$.

② Surjective.

For any $\overline{(U, \varphi)} \in \mathcal{O}_{X,a}$, 任何开集可以拆成有限个$D(f_i)$之并, $U = \bigcup_{i=1}^{s} D(f_i)$
pick an $D(f_i)$, $\varphi|_{D(f_i)} = \frac{g_i}{f_i^n}$. so $\overline{(U, \varphi)} \sim \overline{(D(f_i), \frac{g_i}{f_i^n})}$.
比较构造的map，只需把这里变$D(f_i^n)$
通过补集证明，由$V(f_i) = V(f_i^n)$，得 $D(f_i) = D(f_i^n)$.
于是 $\overline{(U, \varphi)} \sim \overline{(D(f_i^n), \frac{g_i}{f_i^n})}$

③ Injective

$\overline{(D(f), \frac{g}{f})} = 0 \Rightarrow \exists\ a \in V \subset D(f)$, s.t. $\frac{g}{f}\big|_V = 0$.

$\Rightarrow \exists\ a \in V \subset D(f)$, s.t. $g|_V = 0$

$\Rightarrow$ 在 g 前乘上 h，构造 $hg = 0$ on total X

open V 拆成有限 $D(f_i)$ 之并，取 $D(h) \subset V$.

则 $\underset{V外0\ \ V上0}{hg} = 0$ on X.

$\Rightarrow$ 把构造的 hg 写成 $h(g\cdot 1 - 0\cdot ?)$ 的形式

$h(g\cdot 1 - 0\cdot f) = 0$ on X. 故 $\frac{g}{f} = \frac{0}{1} \in A(X)_{I(a)}$.

- Δ Ringed space $(\underbrace{X}_{\text{topo space}}, \underbrace{\mathcal{O}_X}_{\text{sheaf of rings, always called structure sheaf}})$

* An aff. var will aways be considered as a ringed space $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is sheaf of regular functions.

* $U \subseteq X$ will always be considered as a ringed space $(U, \mathcal{O}_X|_U)$.

Δ Mor of ringed space: $\begin{cases} f: X \to Y \ \underline{\text{conti}} \\ \varphi \in \mathcal{O}_Y(U),\ \text{pull-back} \\ \text{要求 } f^*\varphi \in \mathcal{O}_X(f^{-1}(U)) \end{cases}$ $f^*\varphi = \varphi \circ f: f^{-1}(U) \to k$

make sure $f^{-1}(U)$ open so $\mathcal{O}_X(f^{-1}(U))$ well-defined.

This pulling back by f yields a k-alg homo

$f^*: \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U))$, $\varphi \longmapsto f^*\varphi = \varphi \circ f$

- Glueing prop for morphisms: 在 open cover 的每一片上是 mor，则整体是 mor.

$f: X \to Y$ be a map of ringed spaces. $\{U_i\}_{i\in I}$ is an open cover of X s.t. $f|_{U_i}: U_i \to Y$ are mor for all i. Then f is a mor.

Pf: Idea: 证明 f is mor $\Rightarrow$ (1) f is conti. (2) $\forall \varphi \in \mathcal{O}_Y(V)$, $f^*\varphi \in \mathcal{O}_X(f^{-1}(V))$ pullback sections

(1) f conti

$f^{-1}(V) = \bigcup_i U_i \cap f^{-1}(V) = \underbrace{\bigcup_i \underbrace{\underbrace{f|_{U_i}^{-1}}_{\text{conti}}(V)}_{\text{open}}}_{\text{open}}$

(2) f pull-back sections $f|_{U_i}: U_i \to Y$ is a mor

$\forall \varphi \in \mathcal{O}_Y(V)$, we have

$$\underbrace{(f^*\varphi)\big|_{U_i \cap f^{-1}(V)}}_{\text{每一小片上的section}} = (f|_{f^{-1}(V)\cap U_i})^*\varphi \in \mathcal{O}_X((f|_{U_i})^{-1}(V)) = \mathcal{O}_X(f^{-1}(V) \cap U_i)$$

Glueing prop for sheaves: $\exists!\ \underbrace{f^*\varphi}_{\text{拼出整个open set上的section}}|_{f^{-1}(V)} \in \mathcal{O}_X(f^{-1}(V))$

- Mor between aff. var. is an n-tuple of regular functions

X, Y aff. var, $U \overset{\text{open}}{\subseteq} X$. Mor $f: U \to Y$ are exactly the maps of the form

$$f = (\varphi_1, \dots, \varphi_n) : U \longrightarrow Y \subseteq \mathbb{A}^n, \quad x \mapsto (\varphi_1(x), \dots, \varphi_n(x)), \text{ with } \varphi_i \in \mathcal{O}_X(U)$$

In particular, when $Y = \mathbb{A}^1$, $f \in \mathcal{O}_X(U)$.

Pf :

- Cofunctor $A(\cdot)$ is fully faithful

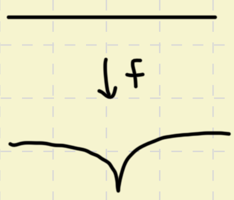
X, Y are aff. vars. There is a bijection

$$\{\text{mor } X \to Y\} \overset{1:1}{\longleftrightarrow} \{k\text{-alg homo } A(Y) \to A(X)\}$$

$$f \longmapsto f^*$$

In particluar, iso $\longmapsto$ iso

• Why we need to consider aff. var as a ringed space (with structure sheaf of regular functions)? why can't we just view them as a topo space?

Exp: 

$X = \mathbb{A}^1$

$\downarrow f$

$Y = V(x_1^2 - x_2^3) \subseteq \mathbb{A}^2$

$f: \mathbb{A}^1 \to Y \quad t \mapsto (t^3, t^2)$ conti

$f^{-1}: Y \to \mathbb{A}^1 \quad (x_1, x_2) \mapsto \begin{cases} \frac{x_1}{x_2} & x_2 \neq 0 \\ 0 & x_2 = 0 \end{cases}$ conti (所有到 $\mathbb{A}^1$ 的 map 都 conti)

→ As Zariski topo space, they are homeomorphic.

这是荒谬的, X没有singular点 而Y有, 因此仅考虑 topo structure 无法区分不同的空间.

f 作为 ringed space mor 不是 iso (这是符合预期的)

若f是 iso 则

$$\varphi: A(V(x_1^2 - x_2^3)) = k[x_1, x_2]/\langle x_1^2 - x_2^3 \rangle \longrightarrow A(\mathbb{A}^1) = k[t]$$ 是 iso

$[P_1: (x_1, x_2) \mapsto x_1] \longmapsto [t \mapsto t^3] =$ poly $t^3 \in k[t]$

$$\boxed{\begin{array}{l} \mathbb{A}^1 \to V(x_1^2 - x_2^3) \xrightarrow{P_1} k \\ t \longmapsto (t^3, t^2) \longmapsto t^3 \end{array}}$$

$[P_2: (x_1, x_2) \mapsto x_2] \longmapsto [t \mapsto t^2]$

$\operatorname{Im}\varphi \overset{\text{set}}{=} k[t^2, t^3] \subsetneq k[t]$, i.e., t 不在 $\operatorname{Im}\varphi$ 中, 即

view function as a poly

Rmk: 涉及多项式与函数map 观点灵活转换.

• △product X×Y 满足 univ. prop

$$\begin{tikzcd} Z \arrow[dr, "\exists!"] \arrow[drr, bend left, "f_X"] \arrow[ddr, bend right, "f_Y"'] & & \\ & X \times Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"] & X \\ & Y & \end{tikzcd}$$

Routine check.

Note: Need to show f we constructed is a mor

Rmk: 不选乘积topo的原因. 正因为X×Y赋予 zariski topo, 才会有这样的 univ. prop.

△ $A(X \times Y) = A(X) \otimes_k A(Y)$

pf: 由 $A(\cdot)$ 是 fully-faith co functor, 上面 univ. prop. 亦有 $A(\cdot)$ 版本, 一望便知.

Rmk: $A(\cdot)$ version 只用在 reduced ring 上 check univ. prop.

• △ Why def of $\underset{\text{new def.}}{\text{aff. var.}}$ is a ringed space iso to $\underset{\text{old def.}}{\text{zero locus of poly in } \mathbb{A}^n}$?

① 这是内蕴的定义（不是靠嵌入 $\mathbb{A}^n$ 来定义的）

② Consider exp.: 嵌入 $\mathbb{A}^n$ 方式不同但同构的例子.（因此定义不应该涉及对 $\mathbb{A}^n$ 的嵌入方式）

Idea：构造同构 aff. var.，只需构造 $A(X) \cong A(Y)$. 我们选定 finitely generated $\underset{\text{No nilpotents}}{\underline{\text{reduced}}}$ k-alg R，设 R 生成元为 $a_1, a_2, \dots, a_n$. 考虑

$$g: k[x_1, \dots, x_n] \longrightarrow R$$
$$f \longmapsto f(a_1, a_2, \dots, a_n)$$

g surj., so $k[x_1, \dots, x_n]/\ker g \cong R$.

$\ker g$ is radical, because R is reduced.

$\left(f^k \in \ker g \Rightarrow f^k(a_1, \dots, a_n) = 0 \overset{R \text{ reduced}}{\Rightarrow} f(a_1, \dots, a_n) = 0 \Rightarrow f \in \ker g\right)$

Then $X = V(\ker g)$ is an aff. var. with $A(X) = k[x_1, \dots, x_n]/\underset{\ker g}{\underbrace{I(X)}} = R$.

Different choice of generators can get different aff. var. X.
But all these X has same coordinate ring so they're all iso.

$R = k[t]$. Choice 1: $R = \langle t \rangle$

$$\rho: k[x_1] \longrightarrow R$$
$$f(x_1) \longmapsto f(t)$$

$\ker \rho = 0$ so $X = V(0) = \mathbb{A}^1$

Choice 2: $R = \langle t, t^2, t^3 \rangle$

$$\rho: k[x_1, x_2, x_3] \longrightarrow R$$
$$f \longmapsto f(t, t^2, t^3)$$

$\ker \rho = \langle x_1^2 - x_2, x_1^3 - x_3 \rangle$

$X = V(\ker \rho) = \underline{\{(x_1, x_1^2, x_1^3) \mid x_1 \in k\}}$

iso to $\mathbb{A}^1$. But it's in $\mathbb{A}^3$!

Different aff. var.

* This example shows

$$\{\text{aff. var.}\}/\text{iso} \overset{1:1}{\longleftrightarrow} \{\text{finitely generated reduced k-alg}\}/\text{iso}$$
$$X \longmapsto A(X)$$
$$V(\ker \rho) \longleftarrow R$$

where $\rho: k[x_1, \dots, x_n] \to R$
$f \longmapsto f(a_1, \dots, a_n)$
$a_1, \dots, a_n$ are generators of R.

△ $A(X) = \mathcal{O}_X(X)$, when we do not embedding X in $\mathbb{A}^n$ in our new def.

△ X aff. var., $f \in A(X)$, $D(f)$ is an aff. var. with coordinate ring

$A(D(f)) = \mathcal{O}_{D(f)}(D(f)) = \mathcal{O}_X(D(f)) = A(X)_f$

In particular, $A(\underset{\mathbb{A}^1}{\underset{\cap}{\underbrace{\mathbb{A}^1 \setminus 0}}}) = A(D(x_1)) = A(\mathbb{A}^1)_{x_1} = k[x_1]_{x_1}$

* 任何 open U 可以被有限 aff. patch $D(f_i)$ cover

pf: Idea: 通过 $\times\mathbb{A}^1$ 增加一维，把 $D(f)\subseteq X$ 嵌入到 $X\times\mathbb{A}^1$ 中写成 zero loci.

Let $Y=V(tf(x)-1)\subseteq \underset{(x,t)}{X\times\mathbb{A}^1}$ $\quad tf(x)-1=0 \Rightarrow t=\frac{1}{f(x)} \Rightarrow Y$ 是 $D(f)$ 上的函数图像 (graph)

$\underset{\text{aff. var.}}{Y}\cong D(f)$ 故 $D(f)$ 是 aff. var. $\quad \Rightarrow Y\cong D(f)$.

$g: Y\to D(f)$, $(x,t)\mapsto x$ $\qquad g^{-1}: D(f)\to Y$, $x\mapsto (x,\frac{1}{f(x)})$

△ $\mathbb{A}^2\setminus 0$ is not aff. Let $U=\mathbb{A}^2\setminus 0$, $X=\mathbb{A}^2$. Assume U aff. var.

$$A(U)=\mathcal{O}_U(U)=\mathcal{O}_X(U)=\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2\setminus 0)\underset{(4)}{=}\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2)=k[x_1,x_2]=A(X)\Rightarrow X\cong U \rightarrow\leftarrow$$

任何 $\mathbb{A}^2\setminus 0$ 上 regular fun 可以延拓到 $\mathbb{A}^2$

- Manifold ⇒ Locally Euclidean
  Prevariety ⇒ Locally affine
  ⇓
  Ringed space with finite open cover by aff. var.s.

  Note: prevariety 的定义只要求存在 finite open aff. cover，而 open cover 本身并不是 prevariety 的 data.

  * Open subset of aff. var. may not be aff. any more, but it can be covered by finite aff. open sets $D(f_i)$, so it's a prevar.

- 把一堆 prevar. 粘起来得到 prevar.

  $\{X_i\}_{i\in I}$ be a set of prevar. We are given $U_{i,j}\subseteq X_i$ for all $i\neq j$ and iso $f_{i,j}: U_{i,j}\to U_{j,i}$ s.t.

  (a) $f_{j,i}=f_{i,j}^{-1}$ $\qquad$ (b) $U_{i,j}\cap f_{i,j}^{-1}(U_{j,k})\subset U_{i,k}$ and $f_{j,k}\circ f_{i,j}=f_{i,k}$ on $U_{i,j}\cap f_{i,j}^{-1}(U_{j,k})$

  $a\overset{(1)}{\sim} f_{ij}(a)\overset{(2)}{\sim} f_{j,i}f_{ij}(a)=f_{ij}^{-1}f_{ij}(a)=a$

  (1)(2): 自反性

  (2): 对称性

  $a\sim f_{ij}(a)$, $f_{ij}(a)\sim f_{jk}f_{ij}(a)$, then $a\sim f_{jk}(a)=f_{jk}f_{ij}(a)$, 传递性满足

  ((a)(b) 只为了说明 $a\sim f_{ij}(a)$ 是等价关系)

  We construct prevariety obtained by glueing X_i along the iso f_{ij} as following:

  (1) As a set, $X=\coprod_{i\in I}X_i/\sim$, $\quad a\sim b \Leftrightarrow \exists f_{ij},\ b=f_{ij}(a)$

  (b) As a topo space, it's defined by quotient topo, i.e., $\iota_i: X_i\to X$

  $U\subseteq X$ open $\Leftrightarrow \iota_i^{-1}(U)$ open in X_i for all $i\in I$

  (c) As a ringed space, $\mathcal{O}_X(U)=\{\varphi: U\to k \mid \iota_i^*\varphi\in\mathcal{O}_{X_i}(\iota_i^{-1}(U))$ for all $i\}$

  Idea: 都是拉回到每一小块上去定义

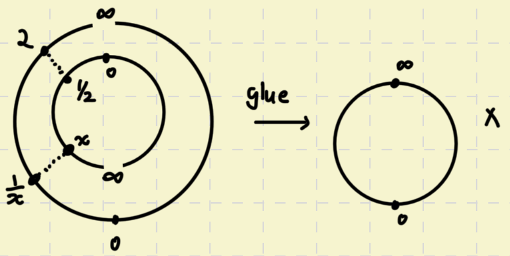
- prevar. is Noetherian，所有 Noetherian space 的结论它都满足.

• EXAMPLE: $X_1 = X_2 = \mathbb{A}^1$ $\quad U_{1,2} = U_{2,1} = \mathbb{A}^1 \setminus 0$

不同 $f: U_{12} \to U_{21}$ 给出不同粘连规则，从而得到不同 X.

Choice 1: $f: U_{12} \longrightarrow U_{21}$, $x \mapsto \frac{1}{x}$

$\underset{\substack{\Uparrow \\ U_1}}{x_1} \sim \underset{\substack{\Uparrow \\ U_2}}{x_2} \Leftrightarrow x_2 = \frac{1}{x_1}$



从图中看 $X = \mathbb{A}^1 \cup \{\infty\}$

$= \mathbb{A}^1 \cup \mathbb{P}^0$

$= \mathbb{P}^1$

More details: ① When $\mathbb{A}^1 = \mathbb{A}^1_{\mathbb{C}}$, $X = \mathbb{C} \cup \{\infty\}$, real pts is a circle.

② $\underset{\substack{\Uparrow \\ U_1}}{x_1} \sim \underset{\substack{\Uparrow \\ U_2}}{\frac{1}{x_1}}$ 实际上在说 $[x_1, 1] \sim [1, \frac{1}{x_1}]$，正是 $\mathbb{P}^1$ 中的等价关系

Application of glueing prop for mors: 构造 $X \to X \overset{\text{glue prop}}{\Leftrightarrow}$ 构造 $\begin{cases} X_1 \to X \\ X_2 \to X \end{cases}$ (可以不满)

$\begin{cases} X_1 \to X_2, & x \mapsto x \\ X_2 \to X_1, & x \mapsto x \end{cases}$ glue to a mor $X \to X$

这个 mor 是 $X \longrightarrow X$, $x \mapsto \frac{1}{x}$ $\quad (\frac{1}{0} = \infty, \frac{1}{\infty} = 0, X = \mathbb{A}^1 \cup \{\infty\})$

$\begin{matrix} X_1 \to X_2 \\ x \mapsto x \end{matrix}$ 实际上是 $X_1 \longrightarrow X_2$

$(x)_{x_1} \mapsto (x)_{x_2} = (\frac{1}{x})_{x_1}$

$(0)_{x_1} \mapsto (0)_{x_2} = \underline{(\infty)_{x_1}}$

因为这么认为

$X_2 \longrightarrow X_1$

$(x)_{x_2} \mapsto (x)_{x_1} = (\frac{1}{x})_{x_2}$

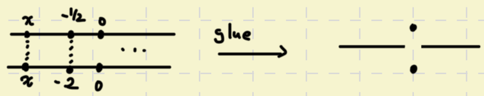
$(0)_{x_2} \mapsto (0)_{x_1}$

$\overset{\shortparallel}{(\infty)_{x_1}}$

两边换到同一坐标

只看 x_1 坐标且记 $(\infty)_{x_1} = \infty$，并认为 $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$，则有 $x \mapsto \frac{1}{x}$

Choice 2: $f: U_{12} \longrightarrow U_{21}$ $\quad x \mapsto x$ be id.



It's an aff. line with double zeros.

$\begin{cases} X_1 \to X_2 \subset X, & x \mapsto x \\ X_2 \to X_1 \subset X, & x \mapsto x \end{cases}$

$X_1 \longrightarrow X_2$

$x \neq 0$ $\quad (x)_{x_1} \longmapsto (x)_{x_2} = (x)_{x_1}$

$(0)_{x_1} \longmapsto (0)_{x_2}$

没有 X_1 中等价类

$X_2 \longrightarrow X_1$

$x \neq 0$ $\quad (x)_{x_2} \longmapsto (x)_{x_1} = (x)_{x_2}$

$(0)_{x_2} \longmapsto (0)_{x_1}$

没有 X_2 中等价类.

构造 $X \to X \overset{\text{glue prop}}{\Leftrightarrow}$ 构造 $\begin{cases} X_1 \to X \\ X_2 \to X \end{cases}$ (可以不满)

此 mor 是交换零点，其余点不变

• What kind of subset in prevariety is prevariety?

① open subsets of prevar is prevar.

$X = \bigcup_{i=1}^{s} X_i$ is aff. open cover. $U \overset{\text{open}}{\subseteq} X$. $\quad U = \bigcup_{i=1}^{s} \underbrace{X_i \cap U}_{\text{open in } X_i \text{ aff.}}$

open subsets of aff. var $X_i \cap U$ is prevar, so U is prevar

(as a finite union of prevar)

$\mathcal{O}_U(V) := \mathcal{O}_X(V)$ is nicely worked, because $V \overset{\text{open}}{\subseteq} U \overset{\text{open}}{\subseteq} X$ means $V \overset{\text{open}}{\subseteq} X$

② Closed subsets of prevar is prevar.

$Y \overset{\text{closed}}{\subseteq} X$. $\quad \mathcal{O}_Y(U) \not\subseteq \mathcal{O}_X(U)$, because $U \overset{\text{open}}{\subseteq} Y \overset{\text{closed}}{\subseteq} X$ doesn't mean $U \overset{\text{open}}{\subseteq} X$. So we define

$\mathcal{O}_Y(U) = \{\varphi: U \to k \mid \forall a \in U\ \exists\ a \in V \overset{\text{open}}{\subseteq} X,\ \psi \in \mathcal{O}_X(V),\ \psi = \varphi \text{ on } U \cap V\}$

A regular fun on an open subset of closed subprevar Y of a prevar X is locally the restriction of a regular fun on X

③ Mix open and closed may not be a prevariety

$X = \mathbb{A}^2$ $\quad U_1 = \underset{\text{open}}{\mathbb{A} \times (\mathbb{A} \setminus 0)}$ $\quad U_2 = \underset{\text{closed}}{\{0\}}$ $\qquad U_1 \cup U_2$

假如它是prevar，则 0 处必存在 aff. var. cover 它.
但 0 附近不存在 aff. var. cover，因为 0 附近显然没有闭子集（极限点不包在里面）

• Products of prevars：

※ 若使用 glueing 来定义 product，过程会非常复杂，而且还要证明所定义的 product 与 glueing 无关. 这时就显出用 univ. prop. 定义 product 的简单了.

△ product of prevar X and Y is a prevar P 满足 univ prop

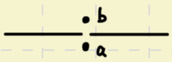
$$\begin{tikzcd} Z \arrow[dr, dashed, "f"] \arrow[drr, bend left, "f_X"] \arrow[ddr, bend right, "f_Y"'] & & \\ & P \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"] & X \\ & Y & \end{tikzcd}$$

△ Existence and uniqueness of products

• Separate：Hausdorff nonsense 的解决办法.

△ Intuition picture：(1) In Classical topo, Hausdorff $\Leftrightarrow \Delta_X \overset{\text{closed}}{\subseteq} X \times X$

In classical topo we use 乘积拓扑

X:  aff. line with double zeros is not sep.

$X \times X$ 有4个 zeroes $(a,b), (b,a), (a,a), (b,b)$

Δ_X 只含 $(a,a), (b,b)$，所以 Δ_X 不closed in $X \times X$

(2) Separate means every sequence has at most one limit

$\underbrace{\lim_{n\to\infty} (x_n, x_n)}_{\Delta_X \text{中一个序列的极限}} = (a,b) \notin \Delta_X$ 说明 Δ_X 不 closed in $X \times X$

△ Def: A prevar X is called a variety (or separated) if the diagonal $\Delta_X \overset{\text{closed}}{\subseteq} \underset{\text{zariski topo}}{X \times X}$

- 一些 var 的例子

(1) aff. vars are vars. $\Delta_X = V(x_1-y_1, x_2-y_2, \cdots, x_n-y_n) \overset{\text{closed}}{\subseteq} X \times X$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ni ((x_1,\cdots,x_n),(y_1,\cdots,y_n))$

(2) Open and closed subprevars are vars, called open and closed subvars.

$$\underset{\substack{\text{open}\\ \text{or}\\ \text{closed}}}{Y} \subseteq \underset{\text{var}}{X} \Rightarrow Y \text{ is prevar} \Rightarrow Y\times Y \text{ prevar} \Rightarrow i: Y\times Y \to X\times X$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cup$ closed

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Delta_X$

$\Longrightarrow \Delta_Y = i^{-1}(\Delta_X)$ is closed $\Longrightarrow$ Y is var.

(3) A var of pure dim 1 is called curve,
of pure dim 2 is called surface.

A pure dim closed subvar $Y \subseteq \underset{\text{pure dim}}{X}$ with $\dim Y = \dim X - 1$,
then Y is called hypersurface

(4) $f, g: \underset{\text{prevar}}{X} \to \underset{\text{var}}{Y}$ mor of prevars.

(4.1) The graph $\Gamma_f := \{(x, f(x)) \mid x \in X\}$ is closed in $X \times Y$

By univ prop of product, there is $(f, id): X\times Y \to Y \times Y$

mor $A \to X\times Y \Leftrightarrow \begin{cases} A \to X \\ A \to Y \end{cases}$ $\qquad\qquad (x,y) \mapsto (f(x), y)$

$\Gamma_f = (f, id)^{-1}(\Delta_Y)$ closed by conti.

(4.2) The set $\{x \in X : f(x) = g(x)\}$ is closed in X

By univ prop of product, $(f,g): X \to Y \times Y$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \mapsto (f(x), g(x))$

The set equals to $(f,g)^{-1}(\Delta_Y)$ is closed in X.

(5) $X, Y \subseteq \mathbb{A}^n$ are pure-dim aff. var., then every irr component of $X \cap Y$ has dim at least $\dim X + \dim Y - n$.

f值不良定! f不是$\mathbb{P}^n$上的函数! 只有f零点良定

$f(\lambda x_0, \cdots \lambda x_n) = 0 \Leftrightarrow f(x_0, \cdots, x_n) = 0$

- Intuitive picture of Projective variety

不齐次 则这个关于齐次坐标的多项式的零点不良定

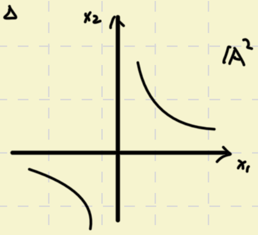
(1) 类比 aff. var $\subseteq$ aff. space $\mathbb{A}^n$, proj var = 齐次多项式的零点 $\subseteq$ proj space $\mathbb{P}^n$

(2) $\mathbb{P}^n$ 是 vect space K^{n+1} 中 1-dim vect spaces 构成的 set. $\mathbb{P}^n = (K^{n+1} \setminus \{0\})/\sim$

(3) $[0,0,0] \notin \mathbb{P}^n$: show a mor well-defined 时必须记 check.
与不能作出直线来

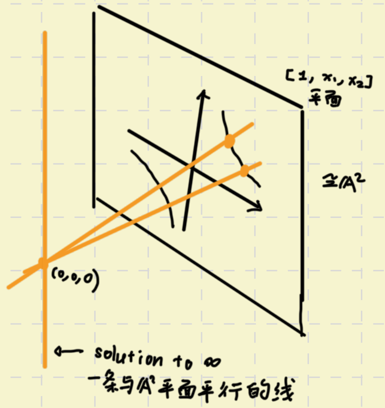
(4) 把 ∞ 添进去, Local 描述: 一堆 aff. var. 做 glueing 得 var.
globally 描述: proj. var.

projective var. 如何 include ∞ pt? Global 描述 $\mathbb{P}^n$ 更简洁



In $\mathbb{A}^2$, solution $x_1x_2-1=0$

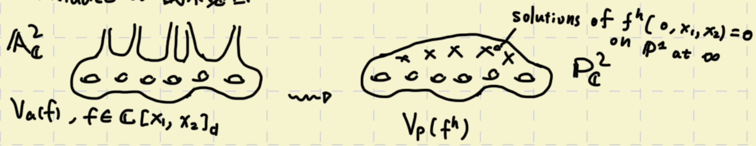
goes to ∞



△ $\mathbb{P}^n = \underbrace{\{(1:x_1:\cdots:x_n)\}}_{\substack{\cong \\ \mathbb{A}^n \\ \text{open}}} \cup \underbrace{\{(0:x_1:\cdots:x_n)\}}_{\substack{\cong \\ \mathbb{P}^{n-1} \\ \text{closed}}} = \underbrace{\mathbb{A}^n}_{\text{aff. part}} \cup \underbrace{\mathbb{P}^{n-1}}_{\text{pts at infinity}}$

open ⟷ closed 互为补集.

△ $\mathbb{P}^n$ includes ∞ 的示意图

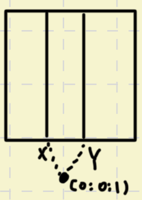


(5) 由于 proj var 添加了 ∞ pt, 在 proj var 中非常容易相交.

prop: $X, Y \subseteq \mathbb{P}^n$ are proj var. then $\boxed{\dim X \cap Y \geq \dim X + \dim Y - n}$

例如 $X, Y \subseteq \mathbb{P}^2$, $\dim X = \dim Y = 1$. $1+1=2$, $\dim X \cap Y \geq 0$ X, Y 相交

假设 X, Y 是在同一个 aff. patch 上的直线



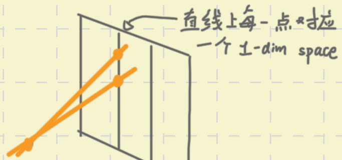
$X = \{(1:0:x_2)\}$

$Y = \{(1:1:x_2)\}$

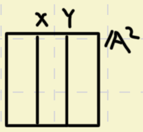
$\lim\limits_{x_2\to\infty}(1:0:x_2) = \lim\limits_{x_2\to\infty}(\frac{1}{x_2}:0:1) = (0:0:1)$

$\lim\limits_{x_2\to\infty}(1:1:x_2) = \lim\limits_{x_2\to\infty}(\frac{1}{x_2}:\frac{1}{x_2}:1) = (0:0:1)$

Rmk: 图中看起来 X, Y 是两条直线, 实际上在 $\mathbb{P}^2$ 中是



It's not true in aff. var. $\mathbb{A}^2$ 中两个 1-dim aff. var. 可以不交.



- Graded ring $\begin{cases} R = \bigoplus\limits_{d\in\mathbb{N}} R_d \Rightarrow f = \sum\limits_d f_d \quad \text{(有限和)} \\ f \in R_d,\ g \in R_e \Rightarrow fg \in R_{d+e} \end{cases}$

  elements of $R_d \setminus \{0\}$ are said to be homogeneous of deg d

  Exp: $k[x_0, \ldots, x_n] = \bigoplus\limits_d k[x_0, \ldots, x_n]_d$

- Homogeneous ideal : special ideals in homogeneous ring

  △ Intuitive picture: It must contain non-homog elements

  since $\underset{\substack{\uparrow \\ \text{ideal } J}}{\text{齐次}} \cdot \underset{\substack{\uparrow \\ R}}{\text{不齐次}} = \text{不齐次} \in J$ e.g. $J = \langle x^2 \rangle \trianglelefteq k[x]$ $x^2(2+x) = \underbrace{2x^2 + x^3}_{\text{non-homog.}} \in J$

  △ Def: An ideal in a graded ring is called homogeneous if it can be generated by homogeneous elements, i.e. $J = \langle f_1, \ldots, f_s \rangle$, f_s homog

- props for homo ideals

(1) The ideal J is homo iff $\forall f \in J$ with homo decomposition $f = \sum_d f_d$ we also have $f_d \in J$ for all d.

pf: $\Rightarrow$ Let $J = \langle h^{(i)} : i \in I \rangle$, $h^{(i)}$ homo $\xRightarrow{f \in J} f = \sum_{i \in I} g^i h_i$ for some $g^i \in R$

$g^i = \sum_{e \in \mathbb{N}} g^i_e$ be homo decomposition of g^i, then degree d part of f is

$$f_d = \sum_{\substack{i \in J \\ e + \deg h^i = d}} g^i_e h^i$$ 由 $\langle h^i \rangle$ 生成 故 $f_d \in J$

$\Leftarrow$ Claim $J = \langle h_d : h \in J, d \in \mathbb{N} \rangle$

"$\subseteq$" $\forall f \in J$, f can generated by f_d.

"$\supseteq$" $h_d \in J$, $\forall h \in J, d \in \mathbb{N}$ by assumption.

(2) J_1, J_2 homo then $J_1 + J_2$, $J_1 J_2$, $J_1 \cap J_2$, $\sqrt{J_1}$ are also homo

Pf: $J_1 + J_2 = \langle J_1 \cup J_2 \rangle$ is also generated by homo elements $\Rightarrow J_1 + J_2$ is homo

(齐次·齐次=齐次) $J_1 J_2 = \langle \underbrace{h_i g_j}_{\text{homo}} : h_i, g_j$ 分别是 J_1, J_2 的齐次生成元$\rangle \Rightarrow J_1 J_2$ is homo

For $f = \sum_d f_d$, $f_d \in J_1$, $f_d \in J_2$ by (1), so $f_d \in J_1 \cap J_2 \Rightarrow J_1 \cap J_2$ is homo

$f = \sum_d f_d \in \sqrt{J_1} \Rightarrow (f_0 + \cdots + f_d)^n = \underset{\text{唯一}d\text{次项}}{\underline{\underline{f_d^n}}} + (\text{terms of lower degree}) \in J_1$

By (1), $f_d^n \in J_1 \Rightarrow f_d \in \sqrt{J_1}$. $f - f_d^n = f_0 + \cdots + f_{d-1}$ lies in $\sqrt{J_1}$ as well,

by induction we have $f_0, f_1, \ldots, f_d \in \sqrt{J_1} \Rightarrow \sqrt{J_1}$ is homo

(3) Graded ring 的商环 (商 homo ideal) 仍然 graded. $R/J = \bigoplus_{d \in \mathbb{N}} R_d / (R_d \cap J)$

Pf: w.t.s. $\forall \bar{f} \in R/J$ $\exists !$ decomposition $\bar{f} = \sum_d \bar{f}_d$, $\bar{f}_d \in R_d \cap J$

存在性由 $f = \sum f_d \in R$ induce, 唯一性: 设 $\bar{f} = \sum_d \bar{f}_d = \sum_d \bar{g}_d$.

$$\overline{\sum (f_d - g_d)} = \sum \bar{f}_d - \sum \bar{g}_d = 0 \Rightarrow \sum (f_d - g_d) \in J \xRightarrow{J \text{ homo}} f_d - g_d \in J \Rightarrow \bar{f}_d = \bar{g}_d$$

-

Δ $V_p(S)$: proj var. $S \subset k[x_0, \ldots, x_n]$ a set of homo polys

$\langle S \rangle = J$ is homo ideal. $V(J) = V(S)$

$\boxed{V_p(J) = \{ x \in \mathbb{P}^n \mid f(x) = 0, \forall \text{ homo } f \in J \}}$

Δ $V_p(f_1, \ldots, f_r) \subseteq \mathbb{P}^n$, $f_1, \ldots, f_r \in k[x_0, \ldots, x_n]$ homo linear poly is a linear subspace of $\mathbb{P}^n$.

Δ $I_p(X) = \langle f \in k[x_0, \ldots, x_n]$ homo $: f(x) = 0 \ \forall x \in X \rangle$ is a homo ideal in $k[x_0, \ldots, x_n]$

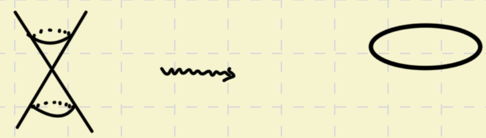
Δ Exp of sub proj. var. (省略下标p)

$Y = V(x_0 + x_1 + x_2) \subseteq \mathbb{P}^2_{\mathbb{C}}$

$$\begin{aligned} V_Y(-x_0^2 + x_1^2 + x_2^2) &= V(x_0 + x_1 + x_2, -x_0^2 + x_1^2 + x_2^2) \\ &= V(x_0 + x_1 + x_2, \underbrace{-(-x_1 - x_2)^2 + x_1^2 + x_2^2}_{= -2x_1 x_2 = 0 \Rightarrow x_1 = 0 \text{ or } x_2 = 0}) \\ &= \{ (-1:0:1), (-1:1:0) \} \end{aligned}$$

- Cone Intuitive picture:

(1) $V_a(x^2+y^2-z^2) \subseteq \mathbb{A}^3_{\mathbb{R}}$ $\qquad\qquad$ $V_p(x^2+y^2-z^2) \subseteq \mathbb{P}^2_{\mathbb{R}}$

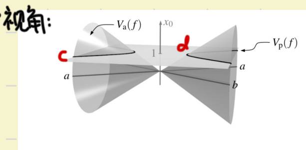


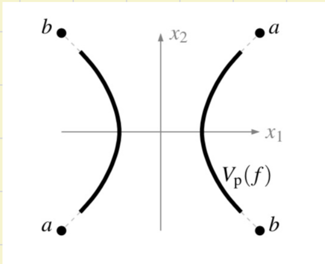
Cone in $\mathbb{A}^3$

Cone $\subseteq \mathbb{A}^{n+1}$ 是 $\mathbb{P}^n$ 中的set每一点长直线得到的

(2) $f = x_1^2 - x_2^2 - x_0^2 \in \mathbb{C}[x_0, x_1, x_2]$

$\mathbb{A}^3$视角:





灰色部分是$\mathbb{A}^3$中的cone

$\mathbb{P}^2$视角: 这个cone上所有直线是$V_p(f)$

c, d是cone上每条直线对应表示

c, d, a, b 合起来是$V_p(f)$. 容易看出

a, b是无穷远点

- Cones and projection

Def: $\mathbb{A}^n \supseteq$ Cone X $\begin{cases} \text{origin} & 0 \in X \\ \text{a union of lines through } 0. & \forall x \in X, \lambda \in k, \ \lambda x \in X \end{cases}$

Def:

$\pi: \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n, \ (x_0, \cdots, x_n) \mapsto (x_0 : \cdots : x_n)$

(1) For a cone $X \subset \mathbb{A}^{n+1}$, we call

$\mathbb{P}(X) := \pi(X \setminus 0) = \{(x_0 : x_1 : \cdots : x_n) \in \mathbb{P}^n \mid (x_0, \cdots, x_n) \in X\} \subseteq \mathbb{P}^n$

the projection of X

(2) For a projective var $Y \subseteq \mathbb{P}^n$, we call

$C(Y) := \{0\} \cup \pi^{-1}(Y) = \{0\} \cup \{(x_0, \cdots, x_n) \mid (x_0 : \cdots : x_n) \in Y\} \subseteq \mathbb{A}^{n+1}$

the cone over Y

- Every cone has the form $V_a(S)$ for some $S \subset k[x_0, \cdots, x_n]$ a set of homo poly

$\left(\begin{array}{l} \text{Every aff. var. has form } V_a(H),\ H \subset k[x_0, \cdots, x_n] \\ \text{Every proj. var. has form } V_p(S),\ S \subset k[x_0, \cdots, x_n] \end{array}\right)$

pf:

$\Leftarrow$ $S \subset k[x_0, \cdots, x_n]$ a set of homo poly, then $V_a(S)$ is a cone

for any $x \in V_a(S) \Rightarrow \forall f \in V_a(S)$ satisfy $f(x) = 0 \Rightarrow \forall \lambda \in k,\ f(\lambda x) = \lambda^{\deg} f(x) = 0, \forall f \in V_a(S) \Rightarrow \lambda x \in V_a(S)$

$\Rightarrow$ Let X be a cone, it's an aff. var so we can use tools in aff. var.

Show $X = V_a(S)$ is equiv to show $I(X) = \langle S \rangle$ is a homo ideal

For any $f \in I(X)$, $f = \sum_{d \in \mathbb{N}} f_d$. w.t.s. $f_d \in I(X)$ (By prop of homo ideal)

X is a cone, so $\forall \lambda \in k, \forall x \in X$, $\lambda x \in X$. By $f \in I(X)$, $f(\lambda x) = 0$.

so $f(\lambda x) = \sum f_d(\lambda x) = \sum \lambda^d f_d(x) = 0$ holds for $\forall \lambda, \forall x \Rightarrow f_d(x) = 0 \ \forall x \in X \Rightarrow f_d \in I(X)$

- prop: $\dim C(X) = \dim X + 1$, $X \subseteq \mathbb{P}^n$ is a proj. var.

- Transform a proj var to an aff. var

(非常多问题要用 $C(\text{proj var})$ 化到 aff. var. 来证明)

Bijection:

$$\{\text{Cones in } \mathbb{A}^{n+1}\} \xleftrightarrow{1:1} \{\text{proj. vars in } \mathbb{P}^n\}$$
$$X \longmapsto \mathbb{P}(X)$$
$$C(Y) \longleftarrow\!\shortmid\; Y$$

Pf: Idea: We've known forms of cones & proj. vars!

$\forall$ cone $X \subseteq \mathbb{A}^{n+1}$, $X = V_a(S)$ for some $S \subset k[x_0,\dots,x_n]$ homo

$$\{\text{Cones}\} \to \{\text{proj var}\} \to \{\text{cones}\}$$
$$V_a(S) \longmapsto \underset{\mathbb{P}(V_a(S))}{V_p(S)} \longmapsto \underset{C(V_p(S))}{V_a(S)}$$

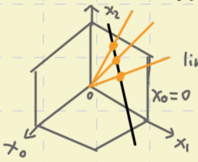
都是 identity

they are bijection

$$\{\text{proj. var}\} \to \{\text{cones}\} \to \{\text{proj. var.}\}$$
$$V_p(S) \longmapsto \underset{C(V_p(S))}{V_a(S)} \longmapsto \underset{\mathbb{P}(V_a(S))}{V_p(S)}$$

* 这个 bijection 的意义在于帮助思考 proj var. 是什么.

例如 a (linear) line in $\mathbb{P}^2$?? equiv to a 2-dim (linear) vec space in $\mathbb{A}^3$



line in $\mathbb{P}^2$: 黄线

$x_0=0$ 平面: a 2-dim vec space in $\mathbb{A}^3$

* 所有 proj var in $\mathbb{P}^n$ 并不神秘, 它就是熟悉的 $\mathbb{A}^{n+1}$ 中的特殊 aff. var. — cone!

- proj version Hilbert Null thm

△ Only difference: (Irrelevant ideal)

Proj Var in $\mathbb{P}^n$ ⇝ Cone in $\mathbb{A}^{n+1}$ $\not\ni 0$ (0 不是 $\mathbb{A}^{n+1}$ 的 cone)

待定义 proj version of Hilbert null thm

aff. version of Hilbert null thm

对应下来有 $I(0)$ & {ideals}

{special ideals in $k[x_0,\dots,x_n]$}

$\ni I(0) = \langle x_0, x_1, \dots, x_n\rangle$

They should correspond to same ideal

因此, 借用 aff. var. 的观点, 我们猜测 $I(0)=\langle x_0,x_1,\dots,x_n\rangle$ 应该不会有 proj. var. 与之对应

引入 def

Def: The radical homo ideal $I_0 = \langle x_0, x_1, \dots, x_n\rangle$ is called the irrelevant ideal.

△ Proj. version Hilbert Null thm

a) For any proj. var. $X \subseteq \mathbb{P}^n$, we have $V_p I_p(X) = X$

b) For any homo ideal $J \trianglelefteq k[x_0,\dots,x_n]$ with $\sqrt{J} \neq I_0$, we have $I_p V_p(J) = \sqrt{J}$

There is a bijection

$$\{\text{proj. var. in } \mathbb{P}^n\} \xleftrightarrow{1:1} \left\{\begin{matrix}\text{homo radical ideals in } k[x_0,\dots,x_n] \\ \text{not equal to irrelevant ideal}\end{matrix}\right\}$$
$$X \longmapsto I_p(X)$$
$$V_p(J) \longmapsto J$$

pf: VERY useful bijection for helping consider proj. var. :

$\mathbb{P}(X) \longleftrightarrow X$

Proj Var in $\mathbb{P}^n$ $\leftarrow$ Cone in $\mathbb{A}^{n+1}$

proj version of Hilbert null thm

aff. version of Hilbert null thm

$\{$special ideals in $k[x_0,\dots,x_n]\}$

They should correspond to same ideal

$I_p(\mathbb{P}(X)) = I_a(X)$

should be same

Claim: (1) $X \neq 0$, $I_p(\mathbb{P}(X)) = I_a(X)$

(2) $\mathbb{P}(V_a(J)) = V_p(J)$ we've proved it

$\mathbb{P}(\phi) = \phi$ obviously

$I_p\mathbb{P} = I_a$

$\mathbb{P}V_a = V_p$

把 I_p, V_p 与 I_a, V_a 关联起来

If the claim is true, then

pf for (a): $\forall X \neq \phi$, $V_pI_p(X) = V_p I_p \overset{\text{无中生有}}{\mathbb{P}C}(X) = V_p I_a C(X) = \mathbb{P}V_a I_a C(X) = \mathbb{P}C(X) = X$

$X = \phi$ $V_pI_p(X) = V_pI_p(\phi) = V_p(k[x_0,\dots,x_n]) = \phi$

pf for (b): $\forall$ homo radical ideal J with $\sqrt{J} \neq I_0$, $I_pV_p(J) = I_p\mathbb{P}V_aJ = I_aV_a(J) = \sqrt{J}$

Finally, pf for claim (1):

"$\subseteq$" $\forall f \in I_p(\mathbb{P}(X))$, $f(x)=0$, $\forall x \in \mathbb{P}(X)$. f is homo, hence $f(\lambda x)=0$, $\forall \lambda \in k$, $\forall x \in \mathbb{P}(X)$.

Besides, cone $X \neq 0$, then $\mathbb{P}(X) \neq \phi$. So $f \neq$ 非0常数. With f homo, $f(0)=0$.

Since $\{\lambda x \mid x \in \mathbb{P}(X), \lambda \in k\} \cup \{0\} = X$, we have $f(x)=0$, $\forall x \in X$. So $f \in I_a(X)$.

"$\supseteq$" $\forall f \in I_a(X) \Rightarrow \begin{cases} f = \sum_{d\in\mathbb{N}} f_d,\ f_d \in I_a(X) & (I_a(X) \text{ is homo ideal}) \\ f(x) = \sum_{d\in\mathbb{N}} f_d(x) = 0,\ \forall x \in X & (f \in I_a(X)) \end{cases}$

$\Rightarrow f_d(x)=0,\ \forall d \in \mathbb{N},\ \forall x \in X \xRightarrow{\mathbb{P}(X)\text{是}X\text{子集}} f_d(\bar{x}) = 0,\ \forall d \in \mathbb{N},\ \bar{x} \in \mathbb{P}(X)$ $(x \in X)$

$\xRightarrow{\text{with } f_d \text{ homo}} f_d \in I_p(\mathbb{P}(X)) \Longrightarrow f = \sum_{d\in\mathbb{N}} f_d \in I_p(\mathbb{P}(X))$

- some differences from aff. case

(1) prop of V_p and I_p are same as V_a and I_a BUT

$I_p(X_1 \cap X_2) = \sqrt{I_p(X_1) + I_p(X_2)} = I_0$ 则说明 $X_1 \cap X_2 = \phi$

e.g. $X_1 = V_p(x_0)$, $X_2 = V_p(x_1)$, then $X_1 \cap X_2 = \phi$ and $\sqrt{I_p(X_1) + I_p(X_2)} = \sqrt{\langle x_0\rangle + \langle x_1\rangle} = \sqrt{\langle x_0, x_1\rangle} = \langle x_0, x_1\rangle$

(2) $Y \subseteq \mathbb{P}^n$, coordinate ring $S(Y) = k[x_0,\dots,x_n]/I_p(Y)$

can NOT be viewed as functions on Y.

(homo coord 变化则函数值变化)

- proj. var. 用 $V_p(J)$ 赋予 zariski topo，立刻可以证明

  $\mathbb{P}^n$ is irr of dim-n

  prop： $\{U_i\}_{i\in I}$ is an open cover of topo space X,
  then $\dim X = \sup\{\dim U_i \mid i\in I\}$

  we have open cover $\mathbb{P}^n = \bigcup_{i=0}^{n} D(x_i)$, $D(x_i)\cong\mathbb{A}^n$, so $\dim D(x_i) = n$.

  Hence by prop $\dim \mathbb{P}^n = n$.

- 下面许多关于 proj. var. 的命题证明都需要新工具：齐次化 & 去齐次化

△ dehomognization(去齐次化)：
$$\begin{cases} \forall \text{ homo } f\in k[x_0,\dots,x_n], \text{ dehomognization } f^i := f(x_0=1) \\ \forall \text{ homo ideal } J \trianglelefteq k[x_0,\dots,x_n], \text{ dehomogenization ideal } J^i := \{f^i : f\in J\} \end{cases}$$

dehomogenization is a ring homo $(fg)^i = f^i g^i$ and $(f+g)^i = f^i + g^i$

△ homogenization (齐次化)：
$$\begin{cases} \forall f\in k[x_0,\dots,x_n], \text{ homogenization } f^h := x_0^d f(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}) \\ \forall J \trianglelefteq k[x_0,\dots,x_n], \text{ homogenization ideal } J^h = \langle f^h : 0\neq f\in J\rangle \end{cases}$$

homogenization is not a ring homo $(fg)^h = f^h g^h$ and $(f+g)^h \neq f^h + g^h$

- $\left.\begin{array}{l} \mathbb{A}^n \cong D(x_0) \hookrightarrow \mathbb{P}^n \text{ 有子空间拓扑} \\ \mathbb{A}^n \text{ 用 } V(J) \text{ 定义 Zariski topo.} \end{array}\right\}$ 二者一样吗？

  Claim： (1) $V_p(J)\cap\mathbb{A}^n = V_a(J^i)$ $\quad J \overset{\text{homo}}{\trianglelefteq} k[x_0,\dots,x_n]$

  (2) $V_a(J) = V_p(J^h)\cap\mathbb{A}^n$ $\quad J \trianglelefteq k[x_0,\dots,x_n]$

  (Note that $\mathbb{A}^n\cong D(x_0)\hookrightarrow\mathbb{P}^n$)

  If the caim is true, then zariski close $V_a(J)$ is also subtopo closed $V_p(J^h)\cap\mathbb{A}^n$. Subtopo closed $V_p(J)\cap\mathbb{A}$ is also zariski closed $V_a(J^i)$. Hence two topo are same.

  pf for (1) $V_p(J)\cap\mathbb{A}^n = \mathbb{P}V_a(J)\cap\mathbb{A}^n = \mathbb{P}V_a(J)\cap \underset{\text{设 } x_0=1}{D(x_0)}$

  $\boxed{\begin{array}{l} I_p\mathbb{P} = I_a \\ \mathbb{P}V_a = V_p \end{array}}$ 把 I_p, V_p 与 I_a, V_a 关联起来

  $$\begin{aligned} &= \{(x_0:\dots:x_n) \mid (x_0,\dots,x_n)\in V_a(J)\}\cap D(x_0) \\ &= \{(1:x_1:\dots:x_n) \mid (1,x_1,\dots,x_n)\in V_a(J)\} \\ &= \{(x_1,\dots,x_n) \mid (x_0,x_1,\dots,x_n)\in V_a(J^i)\} = V_a(J^i) \end{aligned}$$

$$\begin{aligned} V_p(J^h)\cap\mathbb{A}^n &= \{(x_0:\dots:x_n)\in\mathbb{P}^n \mid (x_0:\dots:x_n)\in V_p(J^h)\}\cap D(x_0) \\ &= \{(1:x_1:\dots:x_n)\in\mathbb{P}^n \mid (1:x_1:\dots:x_n)\in V_p(J^h)\} \\ &= \{(1:x_1,\dots,x_n)\in\mathbb{P}^n \mid \forall f(x_1,\dots,x_n)\in J,\ f^h(x_0,x_1,\dots,x_n)\big|_{x_0=1} = 0\} \\ &= \{(x_1,\dots,x_n)\in\mathbb{A}^n \mid \forall f(x_1,\dots,x_n)\in J,\ f^{h^i} = f(x_1,\dots,x_n) = 0\} \\ &= V_a(J) \end{aligned}$$

* Geometric picture of homogenization & dehomogenization?

Why we need these tools?

$\mathbb{P}(\cdot)$ 与 $C(\cdot)$ 给出 {proj var in $\mathbb{P}^n$} 与 {cone in $\mathbb{A}^{n+1}$} 之间的bijection

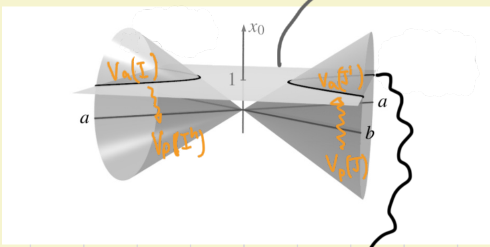
J^h 与 J^i 给出 {proj var in $\mathbb{P}^n$} 与 {aff. var in $\mathbb{A}^n$} 之间的关系 (不一定bijection)

考虑 proj var. 的aff. 截面

aff. part $D(x_0) \cong \mathbb{A}^2$

$V_p(J) \cap \mathbb{A}^n = V_a(J^i)$

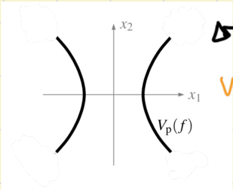
$V_a(J) = V_p(J^h) \cap \mathbb{A}^n$



{proj var in $\mathbb{P}^n$} $\downarrow$ {proj var. 的 aff. part}

$V_p(J) \downarrow V_a(J^i)$

$V_p(I^h) \uparrow V_a(I)$



$V_p(J) \cap \mathbb{A}^n$ or $V_p(I^h) \cap \mathbb{A}^n$

* $J = \langle f_1, f_2, \dots, f_n \rangle$. $J^h \overset{?}{=} \langle f_1^h, f_2^h, \dots, f_n^h \rangle$

这只对主理想正确. $J = \langle f \rangle$, then $J^h = \langle f^h \rangle$

Pf: "$\subseteq$" $J^h = \langle \varphi^h \mid \varphi \in J = \langle f \rangle \rangle$

$= \langle (fg)^h \mid g \in k[x_1, \dots, x_n] \rangle$

$= \langle f^h g^h \mid g \in k[x_1, \dots, x_n] \rangle$

$\subseteq \langle f^h \rangle$

"$\supseteq$" $\forall\, f^h g \in \langle f^h \rangle$, $f \in J$, J^h is an ideal so $f^h g \in J^h$. then $\langle f^h \rangle \subseteq J^h$

• 双曲几何中, 双曲线 & 抛物线 都是圆

抛物线: $J = \langle x_2 - x_1^2 \rangle \trianglelefteq k[x_1, x_2]$ $\quad J^h = \langle x_2 x_0 - x_1^2 \rangle$

$V_p(J^h)$: 一个 $D(x_0)$ 截面是 $V_a(J)$ 双曲线 活在 $\mathbb{P}^2$ 中的 proj. var.

$$V_p(J^h) = \underbrace{V_a(J)}_{x_0=1} \cup \underbrace{[0:0:1]}_{x_0=0}$$

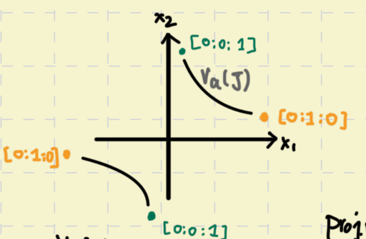
Observe that $\lim_{x_1 \to \infty} \underbrace{[1 : x_1 : x_1^2]}_{\text{双曲线上的点}} = \lim_{x_1 \to \infty} [\frac{1}{x_1^2} : \frac{1}{x_1} : 1] = [0:0:1]$

$$V_p(J^h) = V_a(J) \cup [0:0:1]$$

抛物线在无穷远处被一个点包住, 因此是圆

双曲线: $J = \langle x_1 x_2 - 1 \rangle \trianglelefteq k[x_1, x_2]$ $\quad J^h = \langle x_1 x_2 - x_0^2 \rangle$

$$V_p(J^h) = \underbrace{V_a(J)}_{x_0=1} \cup \underbrace{[0:0:1] \cup [0:1:0]}_{x_0=0}$$



$V_p(J^h)$

$\lim_{x_1 \to \infty} [1 : x_1 : \frac{1}{x_1}] = \lim_{x_1 \to \infty} [\frac{1}{x_1} : 1 : \frac{1}{x_1^2}] = [0:1:0]$

$\lim_{x_2 \to \infty} [1 : \frac{1}{x_2} : x_2] = \lim_{x_2 \to \infty} [\frac{1}{x_2} : \frac{1}{x_2^2} : 1] = [0:0:1]$

Proj var. 中的双曲线在无穷远处包成一个圆

- Another application of homogenization and dehomogenization: Computation of the projective closure

Intuitive picture: (1) 找 proj closure 近似于把射影光源找到并添加进去的过程. 射影光源一般是无穷远点, 找 closure 也可视作添加无穷远点

(2) 上面 $V_a(J) \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$ 并不 closed, 但 $V_p(J^h)$ closed, 就是添加了一些无穷远点

prop: (Computation) $J \trianglelefteq k[x_1, \dots, x_n]$ Consider $X = V_a(J) \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$, compute its closure $\bar{X}$ in $\mathbb{P}^n$.

(a) $\bar{X} = V_p(J^h)$

(b) If $I = \langle f \rangle$, $f \neq 0$, then $\bar{X} = V_p(f^h)$

pf: (a) "$\subseteq$" $X = V_a(J) = V_p(J^h) \cap \mathbb{A}^n \subseteq V_p(J^h)$. $V_p(J^h)$ closed, so $\bar{X} \subseteq V_p(J^h)$

"$\supseteq$" $\bar{X}$ closed, so $\bar{X} = V_p(I)$ for some $I \underset{\text{radical}}{\overset{\text{homo}}{\trianglelefteq}} k[x_0, x_1, \dots, x_n]$

w.t.s. $V_p(J^h) \subseteq V_p(I)$, i.e., w.t.s. $\sqrt{J^h} \supseteq I$.

$\forall g \in I \subseteq k[x_0, x_1, \dots, x_n]$ is an U.F.D., so $g = x_0^s \underset{\text{homo}}{\underline{h}}$ s.t. h 不含 x_0.

于是 h 满足 $(h^i)^h$.

By $\begin{cases} \bar{X} = V_p(I) \\ g \in I \end{cases}$, we have $g(x_0, x_1, \dots, x_n) = 0$, $\forall (x_0, x_1, \dots, x_n) \in \bar{X}$

In particular $g(1, x_1, \dots, x_n) = 0$, $\forall (x_1, \dots, x_n) \in X \subseteq \mathbb{A}^n$,

i.e., $g^i(x_1, \dots, x_n) = 0$, $\forall (x_1, \dots, x_n) \in X$. By $g = x_0^s h$,

$h^i = g^i = 0$ on X, so $h^i \in I(X) = I V_a(J) = \sqrt{J}$.

Then, $\exists\, n \in \mathbb{N}$ s.t. $(h^i)^n \in J$, thus $((h^i)^n)^h \in J^h$

$$((h^i)^n)^h \underset{\substack{\uparrow \\ i \text{ is ring homo}}}{=} ((h^n)^i)^h = h^n \in J^h. \quad \text{So } g^n = x_0^{ns} h^n \in J^h$$

Hence $g \in \sqrt{J^h}$.

(b) 由 $J = \langle f \rangle$ 则 $J^h = \langle f^h \rangle$ 易得.

- X is hypersurface in $\mathbb{P}^n$. $I(X)$ is principal ideal

1^{st} way: $Y := X \cap \mathbb{A}^n$ is hypersurface in $\mathbb{A}^n$. $I(Y) = \langle g \rangle$.

$X = \bar{X} = V_p(I(Y)^h) = V_p(\langle g^h \rangle) \Rightarrow I(X) = \langle g^h \rangle$

2^{nd} way: $I_p(X) = I_a C(X)$ when $X \neq \phi$. $I_p(X) = I_a(C(X)) = \langle f \rangle$

- $\mathrm{Aut}(\mathbb{P}^n) \cong PGL(n+1, k) = GL(n+1, k)/\sim$ $(A \sim \lambda A$ for $\lambda \in k^*)$ (乘法 $[A]\cdot[B] = [A \cdot B]$)

类比于 $\mathrm{Aut}(\mathbb{A}^n) = GL(n, k)$

于是, $\forall a \in \mathbb{P}^n$, $\exists A \in \mathrm{Aut}(\mathbb{P}^n)$, s.t. $Aa = [1 : 0 : \cdots]$ (simplify question) ($\mathrm{Aut}(\mathbb{P}^n)$ 只有线性元)

- proj var.s are prevars. (proj. var. 是有 Global description 的 pre var)

Δ Structure sheaf: $U \underset{\text{proj var}}{\overset{\text{open}}{\subseteq}} X$. A regular fun on U is a map $\varphi: U \to k$ with the following prop: For every $a \in U$, there are $d \in \mathbb{N}$ and $f_a, g_a \in S(X)_d$ with $f(x) \neq 0$ on open $U_a \ni a$, s.t.

$$\varphi(x) = \frac{g_a(x)}{f_a(x)} \text{ on } U_a \subseteq U$$

We construct a sheaf of regular fun (Locally defined, so it's a sheaf)

* 下标d表示 d deg elements in $S(X)$. 上下 deg 相同，使得 φ 真的是一个函数:

$$\frac{f(\lambda x)}{g(\lambda x)} = \frac{\lambda^d f(x)}{\lambda^d g(x)} = \frac{f(x)}{g(x)}$$

* $\mathcal{O}_x(U)$ is a subring of k-alg $\{$function $\varphi: U \to k\}$

△ 要证 $X \subseteq \mathbb{P}^n$ is prevar，只要证 X 有 finite open aff. cover.
显而易见 $X = \bigcup_{i=0}^n (X \cap D(x_i))$ is an open cover of X, it suffices to show $X \cap D(x_i) = \{(x_0 : x_1 : \cdots : x_n) \mid x_i \neq 0\}$ aff.

Let $X = V_p(J)$. 下证 $X \cap D(x_0) =: V_0 \cong V_a(J') =: Y$，其余 $V_1, \dots, V_n$ 同理

$F: Y \longrightarrow V_0 \subseteq \mathbb{P}^n$　　　$F^{-1}: V_0 \longrightarrow Y \subseteq \mathbb{A}^n$　　　对于 $V_0 = X \cap D(x_0)$

$(x_1, \dots, x_n) \longmapsto [1, x_1, \dots, x_n]$　　　$[x_0 : \cdots : x_n] \longmapsto \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$　　　$F^{-1}: V_0 \to Y$

把 $V_a(J') \subseteq \mathbb{A}^n$ 放到 $\mathbb{P}^n$ 的 aff. part $D(x_0)$ 中　　　把 X 的 aff. part V_0 放到 aff. space 中　　　$[1 : x_1 : \cdots : x_n] \mapsto (x_1, \dots, x_n)$

a) Check F, F^{-1} are conti.

For closed set $V_p(J') \cap V_0$ in V_0, $F^{-1}(V_p(J') \cap V_0) = V_a(J'^i)$ closed

For closed set $V_a(I) \cap Y$ in Y, $(F^{-1})^{-1}(V_a(I) \cap Y) = V_p(I^h) \cap V_0$ closed

b) Check pullback.

$\forall \varphi \in \mathcal{O}_{V_0}(U)$, w.t.s. $F^* \varphi \in \mathcal{O}_Y(F^{-1}(U))$　　　$\varphi(x) = \frac{g_a(1 : x_1 : \cdots : x_n)}{f_a(1 : x_1 : \cdots : x_n)}$, $\forall (1 : x_1 : \cdots : x_n) \in U_{F(a)} \subseteq X \cap D(x_0) = V_0$

$\forall a \in F^{-1}(U)$, $\exists$ open $U_{F(a)}$ containing $F(a)$, s.t. $\varphi = \frac{g_a(1 : x_1 : \cdots : x_n)}{f_a(1 : x_1 : \cdots : x_n)}$ on $U_{F(a)}$.

Then on $U_a = F^{-1}(U_{F(a)}) \overset{\text{open}}{\subseteq} F^{-1}(U)$, $F^* \varphi = \frac{g_a^h(x_1, \dots, x_n)}{f_a^h(x_1, \dots, x_n)}$ on U_a

$\forall \psi \in \mathcal{O}_Y(U)$, w.t.s. $F^{-1*}(U) \in \mathcal{O}_{V_0}(F(U))$.

$\forall a \in F(U)$, $\exists$ open $U_{F^{-1}(a)}$ containing $F^{-1}(a)$, s.t.

$$\psi = \frac{g_a(x_1, \dots, x_n)}{f_a(x_1, \dots, x_n)} \text{ on } U_{F^{-1}(a)} .$$

Then on $U_a := F(U_{F^{-1}(a)})$, $F^{-1*}\psi = \frac{g_a(1 : x_1 : \cdots : x_n)}{f_a(1 : x_1 : \cdots : x_n)} = \frac{g_a^h(x_1, \dots, x_n)}{f_a^h(x_1, \dots, x_n)}$ (通过整体乘 x_0^m s.t. $\deg g_a^h = \deg f_a^h$)

• A special type of mors of proj. var.

* In aff. var. case，所有 mor 形如 $(\varphi_1(x), \dots, \varphi_n(x))$, $\varphi_i \in \mathcal{O}_X(U)$.
但在 proj. var. case 没有这样的结论，只有如下更弱的结论.

Let $X \subseteq \mathbb{P}^n$ be proj. var. and $f_0, \dots, f_m \in S(X)_d$. Then on the open subset $U := X \setminus V_p(f_0, \dots, f_m)$, these elements define a mor:

$$f: U \to \mathbb{P}^m, \quad x \longmapsto (f_0(x) : \cdots : f_m(x))$$

* $f_0, \dots, f_m$ have same degree is necessary. $[x_0 : x_1] \longmapsto [x_0^2 : x_1]$ ≠ $[\lambda x_0 : \lambda x_1] \mapsto [\lambda^2 x_0^2 : \lambda x_1]$ =者值不同.

pf: Idea: 用 Glueing prop (Ringed space都有) for mor.

open set $U = X \cap (\bigcap_i V_p(f_i))^c = X \cap (\bigcup_i D_p(f_i)) = \bigcup_{i=1}^m X \cap D_p(f_i) =: \bigcup_{i=1}^m U_i$

U_i open in X. It suffices to show $f|_{U_i}$ is a mor, by glueing prop.

$$f|_{U_i}: X \cap D_p(f_i) \longrightarrow \mathbb{P}^m$$
$$x \longmapsto \left[\frac{f_0(x)}{f_i(x)} : \frac{f_1(x)}{f_i(x)} : \cdots : 1 : \cdots : \frac{f_m(x)}{f_i(x)}\right]$$

观察得 $\begin{cases} \mathrm{Im} f|_{U_i} \subseteq \mathbb{A}^m \hookrightarrow \mathbb{P}^m \\ X\cap D_p(f_i) \subseteq \mathbb{A}^n \end{cases}$ 故 $f|_{U_i}: X\cap D_p(f_i) \longrightarrow \underset{\cup\!\text{l}}{\mathbb{A}^m}$. （化归到 aff. case）

由于 $f|_{U_i}$ 满足 aff. var. 间 mor 的条件，有 $f|_{U_i}$ 是 mor.

$\divideontimes$ 但 Every mor $\mathbb{P}^n \longrightarrow \mathbb{P}^m$ has the form

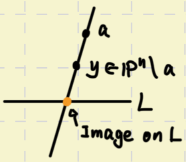
$$f: \mathbb{P}^n \longrightarrow \mathbb{P}^m, \quad x \longmapsto (f_0(x), \cdots, f_m(x))$$

$$f_0, f_1, \cdots, f_m \in k[x_0, \cdots, x_n]_d, \quad V_p(f_0, \cdots, f_m) = \phi$$

- Projection from $a\in\mathbb{P}^n$ to the linear subspace $L \cong \mathbb{P}^{n-1}$

Exp: $a=(1:0:0:\cdots:0)\in\mathbb{P}^n$, $L=V_p(x_0)\cong\mathbb{P}^{n-1}$

则可构造 projection $\mathbb{P}^n\backslash a \to L$ as following



$\forall y=(y_0:y_1:\cdots:y_n)\in\mathbb{P}^n$. l is the line passing a & y.

line l in $\mathbb{P}^n \Leftrightarrow$ 2-dim vect space in $\mathbb{A}^{n+1}$

$l = \{$2-dim vect space 中的点$\}$

$= \{ sa+ty \mid (s,t)\in\mathbb{P}^1 \}$

$= \{ (s+ty_0 : ty_1 : \cdots : ty_n) \mid (s:t)\in\mathbb{P}^1 \}$

Then $l\cap L = l\cap V_p(x_0) = \{ (0: ty_1:\cdots:ty_n) \mid s+ty_0=0,\ (s:t)\in\mathbb{P}^1 \}$

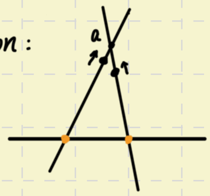
$= \{ (0:y_1:\cdots:y_n) \}$

So projection is $f: \mathbb{P}^n\backslash a \longrightarrow L \cong \mathbb{P}^{n-1}$

$(y_0:\cdots:y_n) \mapsto (0:y_1:\cdots:y_n) \mapsto (y_1:\cdots:y_n)$

Question: 这个 $\mathbb{P}^n\backslash a \longrightarrow \mathbb{P}^{n-1}$ 的 projection 无法延拓到 a .

Reason:



用不同点去逼近 a，得到的像不一样！

Solution: 把 a 放到一个 proj. var. 上，让 projection 限制在这个 proj. var. 上
这样逼近 a 的路线唯一，就可以延拓出 a 处的投影映射了.

Exp: (Extension projection to a) Restrict the projection to a suitable proj. var. $X=V_p(x_0x_2-x_1^2)$

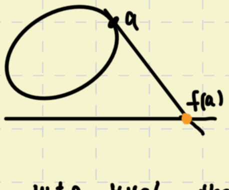
$$f: X \longrightarrow \mathbb{P}^1 \quad (x_0:x_1:x_2) \longmapsto \begin{cases} (x_1:x_2) & \text{if } (x_0:x_1:x_2) \neq (1:0:0) = a \quad \text{(Restriction)} \\ (x_0:x_1) & \text{if } (x_0:x_1:x_2)\neq(0:0:1) \end{cases}$$

打到 $\mathbb{P}^1$ 而不是 L，避免了光源是否在 L 上的讨论

粘的，需让相交部分像相同

$x_0x_2 = x_1^2 \Rightarrow \frac{x_1}{x_2} = \frac{x_0}{x_1}$

f 是两片 mor 粘出来的，因此是 mor，而且在 a 处有定义（几何上是切线与 $\mathbb{P}^1$ 交点）



f is bijective. w.t.s. $\forall y\in L$, the line l passing through a and y intersects X in two points. If it's true, the one is a, the other one is $f^{-1}(y)$. Indeed,

$l = \{ s(0:y_1:y_2) + t(1:0:0) \mid (s:t)\in\mathbb{P}^1 \}$

$= \{ (t: sy_1 : sy_2) \mid (s:t)\in\mathbb{P}^1 \}$

pts in $l\cap X$ fulfilling $tsy_2 - s^2y_1^2 = 0$
there are only two solutions (1) $s=0 \rightsquigarrow a$
(2) $\frac{s}{t} = \frac{y_2}{y_1^2} \rightsquigarrow f^{-1}(y)$

Moreover, f is an iso:

$$f^{-1}: \mathbb{P}^1 \longrightarrow X$$
$$(y_1:y_2) \longmapsto (y_1^2 : y_1y_2 : y_2^2) \quad (\text{用 } \tfrac{s}{t} = \tfrac{y_2}{y_1^2})$$

f^{-1}每个分量是2次poly，故是mor. 于是 f是iso.

* 不是所有proj是iso. $X = V_p(x_0^2 - x_1x_2) \subseteq \mathbb{P}^2$
$(1:0:0) \in X$ 于是可定义 忘掉x_0的mor. $\varphi: X \to \mathbb{P}^1$ $(x_0:x_1:x_2) \mapsto (x_1:x_2)$ well-def.
we find $\varphi(1:1:1) = \varphi(-1:1:1)$.

- Segre embedding $\mathbb{P}^n \times \mathbb{P}^m$ 同构到 $\mathbb{P}^N$ 中一个proj. var.，描述更容易.

$f: \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N$, $N = (n+1)(m+1)-1$
$([x_0:x_1:\cdots:x_n], [y_0:\cdots:y_m]) \longmapsto (x_iy_j) = (z_{ij}) = (x_0y_0, x_0y_1, \cdots, x_0y_m, x_1y_m, \cdots, x_ny_0, \cdots, x_ny_m)$

* Segre embedding is well-defined
$\vec{x} \neq 0$, $\exists x_i = 1$. $\vec{y} \neq 0$, $\exists y_j = 1$. So $z_{ij} = 1$, $(z_{ij}) \neq 0$.

idea (1)证像集是想要的形式 (2)证mor是iso

Δ The image $X = f(\mathbb{P}^n \times \mathbb{P}^m)$ is a proj. var.

$S := V_p(z_{ij}z_{kl} - z_{il}z_{kj} : 0 \le i,k \le n,\ 0 \le j, l \le m$
i,k不动，j,l互换. 因此 $j=l$或$i=k$ 则式子恒为0

pf: $X \subseteq S$: $z_{ij} = x_iy_j$，显然有 $\underbrace{x_iy_j}_{z_{ij}} \underbrace{x_ky_l}_{z_{kl}} - \underbrace{x_iy_l}_{z_{il}} \underbrace{x_ky_j}_{z_{kj}} = 0$

$S \subseteq X$: $\forall z = [z_{00}, \cdots, z_{mn}] \in S$
只要证 z是像. 只要把 $\vec{x}$和$\vec{y}$找到. Assume $z_{00} \neq 0$ (假设非0然后要除掉它)
由 $z \in S = V(z_{ij}z_{kl} - z_{il}z_{kj})$ 得 $z_{00}z_{kl} - z_{0l}z_{k0} = 0$.
两边同除 z_{00}^2 得 $\underbrace{\frac{z_{kl}}{z_{00}}}_{\text{坐标}z_{kl}} = \underbrace{\frac{z_{0l}}{z_{00}}}_{x_l} \underbrace{\frac{z_{k0}}{z_{00}}}_{y_k}$ 令 $x_l = \frac{z_{0l}}{z_{00}}$, $y_k = \frac{z_{k0}}{z_{00}}$

$x = (\underbrace{1}_{x_0 = \frac{z_{00}}{z_{00}} = 1} : x_1 : \cdots : x_n)$, $y = (1 : y_1 : \cdots : y_n)$

$f(x,y) = (x_iy_j) = \frac{z_{ij}}{z_{00}}$ 和预期的只差一个整体的系数

Δ $f: \mathbb{P}^n \times \mathbb{P}^m \longrightarrow X$ is an iso

pf: f是bijection. $\forall z \in X$，不妨设 $z_{00} = 1$，则 $x_0 \neq 0$, $y_0 \neq 0$ 即 $x_0 = y_0 = 1$.
由 $z_{k0} = x_ky_0 = x_k$, $z_{0l} = x_0y_l = y_l$ 可知原像 $\vec{x}, \vec{y}$ 被我们暴力解完了，因此原像唯一. 原像存在(proceeding process)且唯一，即 f是 bijection.
还需证 f&f^{-1}是mor. 不是用 glueing prop，比 section 拉回是section 简单. 用glueing prop 最好拆成aff. parts，因为aff. parts上 mor的形式比 proj. var. 要更清楚.

$U_i := D_p(x_i) \subseteq \mathbb{P}^n$, $V_j = D_p(x_j) \subseteq \mathbb{P}^m$，则 $U_i \times V_j$ are aff. open cover of $\mathbb{P}^n \times \mathbb{P}^m$ (非常常见的拆法)
(别证 $f|_{U_i \times V_j}$，写起来太繁琐，证 $f|_{U_0 \times V_0}$ 上是mor 即可!)

$$f|_{U_0\times V_0}([1:x_1:\cdots:x_n],[1:y_1:\cdots:y_n]) = (1:y_1:\cdots:y_n:x_1:x_1y_1:\cdots:x_ny_n)$$

$\underline{U_0\times V_0 \subseteq \mathbb{A}^n\times\mathbb{A}^m \to D(z_{00})\cong\mathbb{A}^N}$ 每个分量是 $(x_1,\cdots,x_n,y_1,\cdots,y_n)$ 的 regular fun.

同理 $D(z_{ij})\cap X$ 是 X 上 aff. open cover.

$$f^{-1}|_{D(z_{00})}(z_{ij}) = ([1:z_{10}:\cdots:z_{n0}],[1:z_{01}:\cdots:z_{0m}])$$

aff. 间 map 每个分量是 z_{ij} regular fun.

- Segre embedding application I: Every proj var. is a var.

pf: diagonal 出现 $\mathbb{P}^n\times\mathbb{P}^n$，所以用 $[z_{ij}]$ 坐标会很方便.

$$\Delta_{\mathbb{P}^n} = \{((x_0:\cdots:x_n),(y_0:\cdots:y_n)) \mid (x_0:\cdots:x_n)=(y_0:\cdots:y_n)\}$$

☆ $(x_0:\cdots:x_n)=(y_0:\cdots:y_n)$ 的化为 ⟶ MATRIX $\begin{cases}\text{RANK}\\ \text{Det}\end{cases}$

$\Updownarrow$

$$\operatorname{rank}\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ y_0 & y_1 & \cdots & y_n\end{pmatrix} = 1 \iff \text{任何 } 2\times 2 \text{ minors vanish}$$

$$\iff x_iy_j - x_jy_i = 0 \quad \forall i,j$$

$$\iff z_{ij} - z_{ji} = 0$$

$V(z_{ij}-z_{ji} : \forall i,j)$ closed in $\mathbb{P}^{(n+1)^2-1}$，自然 closed in $\mathbb{P}^n\times\mathbb{P}^n = V(z_{ij}z_{kl} - z_{il}z_{kj} : \forall i,j,k,l)$.

Rmk: (VERY USEFUL) $M\in M_{n\times m}(k)$.

$\operatorname{rank}(M)\le k \iff$ 任意 $(k+1)\times(k+1)$ minors vanish.

$\operatorname{rank} M \geq k \iff \exists\ k\times k$-minor $\neq 0$

$\operatorname{rank} M = k \iff \exists\ k\times k$-minor $\neq 0$, $\forall\ (k+1)\times(k+1)$-minor $=0$

- $X\subset\mathbb{P}^m$, $Y\subset\mathbb{P}^n$ are proj. var. then $X\times Y$ is proj. var.

pf: X, Y closed in $\mathbb{P}^m$, $\mathbb{P}^n$, resp. then $X\times Y$ closed in $\mathbb{P}^m\times\mathbb{P}^n$. $\mathbb{P}^m\times\mathbb{P}^n$ is proj. var. closed in $\mathbb{P}^{(m+1)(n+1)-1}$, so $X\times Y$ is also closed in $\mathbb{P}^{(m+1)(n+1)-1}$ and thus $X\times Y$ is a proj. var.

- Closed maps

Motivation: Noetherian spaces are all compact, hence $\mathbb{P}^n_{\mathbb{C}}$ and $\mathbb{A}^n_{\mathbb{C}}$ are compact under zariski-topo. BUT $\mathbb{P}^n_{\mathbb{C}}$ is compact and $\mathbb{A}^n_{\mathbb{C}}$ is uncompact in classical topo. 所以需要引入新概念区分紧性.

<u>Def</u>: A map $f: X\to Y$ between topo space is called closed if $f(A)\subseteq Y$ closed for any closed $A\subseteq X$.

- <u>Prop</u>: projection map $\pi: \mathbb{P}^n\times\mathbb{P}^m \longrightarrow \mathbb{P}^m$ is closed.

* 需要用到结论: For $V_p(f_1,\dots,f_n)$ f_i homo, 总可将 f_i 造成齐相同次的. Key point: Check $V_p(f) = V_p(x_0^sf, x_1^sf, \dots, x_n^sf)$

$$\text{R.H.S.} = \bigcap_i V_p(x_i^sf) = \bigcap_i (V_p(x_i^s)\cup V_p(f)) = \big(\underbrace{\bigcap_i V_p(x_i^s)}_{\emptyset}\big)\cup V_p(f) = V_p(f)$$

e.g. $V_p(f_1,f_2)$, $\deg f_1 < \deg f_2$. Let $s = \deg f_2 - \deg f_1$.

$V_p(f_1,f_2) = V_p(x_0^sf_1, x_1^sf_1, \dots, x_n^sf_1, f_2)$

pf: Idea: 把某个条件用代数几何的语言描述出来

Any closed set X in $\mathbb{P}^n\times\mathbb{P}^m$ can be written as $X=V(f_1,\cdots,f_r)$ where f_i are homo poly in segre coordinates (z_{ij}) with same degree (上面已证总能办到)

(一个关于 Segre coord 的 poly 视为 (x_i) or (y_i) 的 poly degree 相同，因为出现一次 x_i 必出现一次 y_j.)

Let a be a pt.

$a=(a_0:\cdots:a_m)\in\pi(X)\Leftrightarrow\exists(x,y)\in V(f_1,\cdots,f_r)$, s.t. $\pi(x,y)=y=a$

$\Leftrightarrow\exists x\ f_i(x,a)=0,\ \forall i$.

令 $g_i=f_i(\cdot,a)$. 则 g_i 是关于 x 的 poly, $\deg g=\deg f_i$ over $x=:d$

$a\in\pi(X)\Leftrightarrow\exists x,\ g_i(x)=0,\forall i\Leftrightarrow V(g_1,\cdots,g_r)\neq\emptyset$

trick: $V(\cdots)\neq\emptyset$ 不好用，$V(\cdots)=\emptyset$ 才可以用 Hilbert null thm. 因此换成如下等价形式:

$a\notin\pi(X)\Leftrightarrow V(g_1,\cdots,g_r)=\emptyset\Leftrightarrow\sqrt{\langle g_1,\cdots,g_r\rangle}=\langle 1\rangle$ (用 $\langle 1\rangle$ 表示全空间)

or $\sqrt{\langle g_1,\cdots,g_r\rangle}=\langle x_0,\cdots,x_n\rangle$ (relavent ideal)

$\Leftrightarrow\exists k_i\in\mathbb{N}$ s.t. $x_i^{k_i}\in\langle g_1,\cdots,g_r\rangle$ for all i

$\Leftrightarrow\exists k$ (取 $k=k_1+k_2+\cdots+k_r$), s.t. $k[x_0,\cdots,x_n]_k\subseteq\langle g_1,\cdots,g_r\rangle$

$\Leftrightarrow\exists k,\ k[x_0,\cdots,x_n]_k=\langle g_1,\cdots,g_r\rangle_k$ (因为 $\langle g_1,\cdots,g_r\rangle_k\subseteq k[x_0,\cdots,x_n]_k$.)

$$\Leftrightarrow\ \varphi:\underbrace{k[x_0,\cdots,x_n]_{k-d}\times\cdots\times k[x_0\cdots x_n]_{k-d}}_{r\text{个相乘}}\longrightarrow k[x_0,\cdots,x_n]_k$$

(视作 k-vect space)

$$(h_1,\cdots,h_r)\longmapsto\sum \underset{\deg d}{g_i}\underset{\deg k-d}{h_i}$$

is surjective as a k-linear map between vect space

(空间的基是每个分量放单项式 $\dim k[x_0,\cdots,x_n]_k=\binom{n+k}{k}$

$\dim(k[x_0,\cdots,x_n]_{k-d}\times\cdots\times k[x_0,\cdots,x_n]_{k-d})=r\binom{n+k-d}{k-d}$)

只要写出 linear map 矩阵就知道 φ 是否 surj.

$a\notin\pi(X)\Leftrightarrow$ matrix of φ has rank $\binom{n+k}{k}$ (满秩)

$\pi(X)$的补集 $\Leftrightarrow\exists$ minor of $\binom{n+k}{k}\times\binom{n+k}{k}$ with determinate $\neq0$

矩阵的每一个 entry 是 $a_0,\cdots,a_m$ 的 poly with deg d.

e.g. 取左边一个特殊的基 $(x_0^{k-d},0,0,\cdots,0)$

$\varphi(x_0^{k-d},0,\cdots,0)=g_0x_0^{k-d}=f_0(\cdot,a)\,x_0^{k-d}$

是与 a 有关的多项式.

$\Leftrightarrow\det(a_0,\cdots,a_m)\neq0$ 是开条件

- Prop: The projection map $\pi:\mathbb{P}^n\times Y\longrightarrow Y$ is closed for any var. Y

pf: 分步证明 (aff 以及 var 拆成 aff.)

① 假设 $Y\subseteq\mathbb{A}^n$ is aff. 设 $Z\overset{\text{closed}}{\subseteq}\mathbb{P}^n\times Y$

Idea: aff. var. 嵌入到 $\mathbb{P}^m$ 中从而用 $\pi:\mathbb{P}^n\times\mathbb{P}^m\longrightarrow\mathbb{P}^m$ 是 closed 来证.

$$Z\overset{\text{closed}}{\subseteq}\mathbb{P}^n\times Y\subseteq\mathbb{P}^n\times\mathbb{A}^m\subseteq\underset{Z\text{不是这里的闭集}}{\mathbb{P}^n\times\mathbb{P}^m}\xrightarrow{\pi'}\mathbb{P}^m$$

自己制造闭集. $\bar Z$ is closure of Z in $\mathbb{P}^n\times\mathbb{P}^m$. 则 $\pi'(\bar Z)$ closed.

由 $Z=\bar{Z}\cap \mathbb{P}^n\times Y$ （$A\subseteq B\subseteq X$，$\bar{A}_B=\bar{A}_X\cap B$ 包含A的最小B中闭集），VERY USEFUL

得 $\pi(Z)=\pi(\bar{Z}\cap\mathbb{P}^n\times Y)=\pi'(\bar{Z})\cap Y$ 是 Y 中的闭集

$\pi:\mathbb{P}^n\times Y\to Y$

$\pi':\mathbb{P}^n\times\mathbb{P}^m\to\mathbb{P}^m$

② 假设 Y is not aff. $Y=\bigcup_i Y_i$，Y_i aff.

$\forall Z$ closed in $\mathbb{P}^n\times Y$, then $Z\cap\mathbb{P}^n\times Y_i$ is closed in $\mathbb{P}^n\times Y_i$

so $\pi(Z\cap\mathbb{P}^n\times Y_i)$ is closed in Y_i. Then

$\pi(Z)\cap Y_i=\pi(Z)\cap\pi(\mathbb{P}^n\times Y_i)=\pi(Z\cap\mathbb{P}^n\times Y_i)$ is closed in Y_i

~~Hence $\pi(Z)=\bigcup_i\pi(Z)\cap Y_i$ closed in Y~~ ✗

Hence $Y\setminus\pi(Z)=\bigcup_i \underbrace{Y_i\setminus(\pi(Z)\cap Y_i)}_{\text{open in } Y_i}$ $\Rightarrow$ open in Y $\Rightarrow\pi(Z)$ closed.

Y_i open in Y, so it's open in Y.

- Complete var: A var is called **complete** if the projection $\pi:X\times Y\to Y$ is closed for any var Y.

Exp (1) $\mathbb{P}^n$ is complete.

(2) In general topo, 紧空间闭子集是紧的.
In Zariski topo, complete space 的闭子集是 complete.

$Z\underset{\text{closed}}{\subseteq}X_{\text{complete}}$ w.t.s. $\pi:Z\times Y\to Y$ closed.

$\pi: Z\times Y\xrightarrow{\iota}X\times Y\xrightarrow{\pi_X}Y$, $\pi=\pi_X\circ\iota$. π_X is closed, ι is closed, so π is closed. (Why ι closed? $Z\underset{\text{closed}}{\subseteq}X$, $Y\underset{\text{closed}}{\subseteq}Y$ so $Z\times Y\underset{\text{closed}}{\subseteq}X\times Y$ then any closed subset of $Z\times Y$ is still closed in $X\times Y$)

(3) Every proj. var. is complete (complete空间闭子集 complete), BUT NOT all complete var is proj. var.

(4) A complex variety is compact in classical topo iff it's complete

(5) $\mathbb{A}^1$ is not complete (图像上显然，$\mathbb{A}^1$ 没有无穷远点，)

$\pi:\mathbb{A}^1\times\mathbb{A}^1\to\mathbb{A}^1$ $\pi(V(xy-1))=\mathbb{A}^1\setminus\{0\}$ is not closed

$\mathbb{A}^1\times\mathbb{A}^1=\mathbb{A}^2$

$V(xy-1)$ closed in $\mathbb{A}^1\times\mathbb{A}^1$ 把 $\mathbb{A}^2$ 视为 $\mathbb{P}^2$ aff. patch 就显然了.

- Prop: $f:X\to Y$ is mor of var, X is complete **then** $f(X)$ is ~~complete~~ **closed** in Y. (In general topo, 紧集的像是紧的)

* 在 Zariski topo 下，$f(X)$ 不仅 complete，而且 **closed**！

(1) Show $f(X)$ closed.

$X\xrightarrow{(id,f)}X\times Y\xrightarrow{\pi}Y$ $\Gamma_f=\{(x,f(x))\in X\times Y\}$ is **closed** in $X\times Y$

$x\mapsto(x,f(x))\mapsto f(x)$ so $f(X)$ is closed in Y by π closed

(2) Show $f(X)$ complete, i.e., check $f(X)\times Z\to Z$ closed for $\forall Z$.

显然有交换图，把一个未知的mor $f(X)\times Z \to Z$ 与已知性质的mor $X\times Z \to Z$ 联系起来

$$\begin{tikzcd} (x,z) \arrow[r, mapsto] & (f(x),z) \\ X\times Z \arrow[r, "(f,id)"] \arrow[dr, "\pi'"'] & f(X)\times Z \arrow[d, "\pi"] \\ & Z \\ & z \end{tikzcd}$$

$\forall$ closed $W \subseteq f(X)\times Z$

$$\underbrace{\pi(W) = \underbrace{\pi'\underbrace{(f,id)^{-1}(W)}_{\text{closed}}}_{\text{closed}}}_{\text{closed } (X \text{ complete})}$$

- Prop: X: connected complete var. Then $\mathcal{O}_X(X) = k$

pf: $\forall f \in \mathcal{O}_X(X)$ has the form $f: X \to \mathbb{A}^1$.
X complete so $f(X)$ closed in $\mathbb{A}^1$. Closed sets in $\mathbb{A}^1$ has the form {finite pts}. so $f(X)$ is finite pts. X connected so $f(X)$ connected, hence $f(X)$ is a singlet, i.e., f is a const.

- Veronese embedding 把小 proj space 嵌到大 proj space.
pick $n, d \in \mathbb{N}_{>0}$.

Veronese embedding
$$F: \mathbb{P}^n \longrightarrow \mathbb{P}^N$$
$$x \longmapsto (f_0(x) : \cdots : f_N(x))$$

where $f_0, \dots, f_N \in k[x_0, \dots, x_n]$ 是所有 deg d monomials, 共有 $\binom{n+d}{n}$ 个.
由于是从0开始计数, $N = \binom{n+d}{n} - 1$

* Well-defined: $f_0, \dots, f_N$ 包含 $x_0^d, \dots, x_n^d$, 因此无公共零点

* It's a mor: 每个分量是齐d次多项式

* $F(\mathbb{P}^n)$ is a proj. var. in $\mathbb{P}^N$: $\mathbb{P}^n$ complete, so image of mor $F(\mathbb{P}^n) =: X$ closed in $\mathbb{P}^N$ thus it's proj. var in $\mathbb{P}^N$.

*: $F: \mathbb{P}^n \longrightarrow X$ is an iso

Key point: 找到 F^{-1} 再证 F 与 F^{-1} 在每个 aff. patch 上是 mor (routine)

Consider open subset $D(x_i)$ of X, i.e., on this patch X has the form
$$\left(\frac{f_0}{x_i^d} : \frac{f_1}{x_i^d} : \cdots : \underset{i\text{-th}}{1} : \cdots : \frac{f_m}{x_i^d}\right)$$

So $\underbrace{x_j}_{\mathbb{P}^n \text{ coord}} = \dfrac{Z_{(d-1)e_i + e_j}}{Z_{de_i}} = \dfrac{x_j x_i^{d-1}}{x_i^d}$

* e.g. $\mathbb{P}^1 \to \mathbb{P}^{\binom{1+d}{d}-1} = \mathbb{P}^d$
$(x_0 : x_1) \longmapsto (x_0^d : x_0^{d-1}x_1 : \cdots : x_1^d)$

* Importance of Veronese embedding:

Coord in $\mathbb{P}^n$		Coord in $\mathbb{P}^N$
degree-d poly	$\rightsquigarrow$	linear poly

↓example

- Prop: Let $X \subset \mathbb{P}^n$ be a proj var. and $f \in S(X)$ be homogeneous and non constant. Then $X \setminus V(f)$ is an aff. var.

* $\mathbb{P}^n$ 挖掉 hypersurface 都是 aff.

pf: Let $F: \mathbb{P}^n \to \mathbb{P}^N$ be degree d Veronese map e.g. $V(a_0 x_0^d + \cdots + a_n x_n^d)$, 则它在 F 下的像是 $V(a_0 Z_{de_0} + \cdots + a_n Z_{de_n})$ 这是 $\mathbb{P}^N$ 中 hyperplane, Veronese embedding 的好处在于把方程变线性.

即 Veronese embedding 把 deg d hyperplane map 到 $\underset{\text{deg 1}}{}$ (linear) deg 1 hyperplane.
在复合 $\mathrm{Aut}(\mathbb{P}^n)$ 中元素后, 总可假设 map 后的 hyperplane 满足 $z_0=0$.
于是 $F(\mathbb{P}^n)\setminus$hyperplane $\subseteq D(z_0)$ is an aff.

- Grassmanians $G(k,n)$: k-dim linear spaces in K^n

Goal : Show $G(k,n)$ is a proj. var.

Idea : find embedding $G(k,n)\longrightarrow \mathbb{P}^?$

To construct this embedding we introduce alternating tensor products

- Alternating tensor product

Def : V is a K-vector space. $k\in\mathbb{N}$. A k-fold multilinear map $f: V^k\longrightarrow W$ is called alternating if any $v_i=v_j$ for some $i\neq j$ we have $f(v_1,\cdots,v_k)=0$

* 任何两个分量相同会被打到0.
* 等价定义, 一次对换多一个负号 $f(\cdots,v_j,\cdots,v_i,\cdots)=-f(\cdots,v_i,\cdots,v_j,\cdots)$

i.e., $f(v_{\sigma(1)},\cdots,v_{\sigma(k)})=\mathrm{sgn}(\sigma)\,f(v_1,\cdots,v_k)$

Def : (Alternating tensor product)

a k-fold tensor product of V is a vect space T with an k-fold alternating multilinear map $\tau: V^k\to T$ satisfying univ. prop:

$$\begin{tikzcd} V^k \arrow[r, "f"', "\text{alt. multilinear}"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\circlearrowright"] & W \\ T \arrow[ur, dotted, "\exists! g"'] & \end{tikzcd}$$

* The image $\tau(v_1,\cdots,v_k)=: v_1\wedge v_2\wedge\cdots\wedge v_k$
* alt. tens. prod. of V exists, 可以证明 $T=\Lambda^k V$, $\Lambda^k V$ 是以

$\{\,1\le j_1<j_2<\cdots<j_k\le n \quad v_{j_1}\wedge v_{j_2}\wedge\cdots\wedge v_{j_k}\}$ 为基的 Linear space ,
其中 $\{v_i\}_{i\in[\![1,n]\!]}$ 是 V 的 basis.

* $\dim\Lambda^k V=\binom{n}{k}$

* $\Lambda^k V=\otimes^k V/L$, L is vector subspace of $V^{\otimes k}$ generated by $\langle v_1\otimes\cdots\otimes v_k \mid v_i=v_j \text{ for some } i\neq j\rangle$

* $\Lambda^0 V\cong K \quad \Lambda^1 V\cong V$

- Observation: $\dim\Lambda^n K^n=\binom{n}{n}=1$ so $\Lambda^n K^n\cong K$
$\dim\Lambda^2 K^3=\binom{3}{2}=3$ so $\Lambda^2 K^3\cong K^3$.

其中 $\Lambda^n K^n\cong K$ given by determinant

任何 $\Lambda^n K^n$ 中元 $v_1\wedge v_2\wedge\cdots\wedge v_n=\det[v_1\cdots v_n]\,e_1\wedge\cdots\wedge e_n$

$\Lambda^2 K^3\cong K^3$ given by cross product

任何 $\wedge k^3$ 中元素 $v_1\wedge v_2 = (a_1e_1+a_2e_2+a_3e_3)\wedge(b_1e_1+b_2e_2+b_3e_3)$

$$= \underline{(a_1b_2-a_2b_1)}\,e_1\wedge e_2 + \underline{(a_1b_3-a_3b_1)}\,e_1\wedge e_3 + \underline{(a_2b_3-a_3b_2)}\,e_2\wedge e_3$$

系数来自于v_1与v_2组成矩阵的2×2-minor

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

从这个observation出发.

prop: Let $v_1, v_2, \dots, v_k \in k^n$. Assume $v_i = \sum_{j=1}^{n} a_{ij} e_j$

Then $v_1\wedge v_2\wedge\cdots\wedge v_k = \sum$ 对应余子式 $e_{j_1}\wedge\cdots\wedge e_{j_k}$

pf:

$$v_1\wedge v_2\wedge\cdots\wedge v_k = (a_{11}e_1+\cdots+a_{1n}e_n)\wedge(a_{21}e_1+\cdots+a_{2n}e_n)\wedge\cdots\wedge(a_{k1}e_1+\cdots+a_{kn}e_n)$$

$$= \sum a_{1j_1}a_{2j_2}\cdots a_{kj_k}\, e_{j_1}\wedge e_{j_2}\wedge\cdots\wedge e_{j_k}$$

只要对互不相同项求和

$$= \sum_{1\le j_1\ne j_2\ne\cdots\ne j_k\le n} a_{1j_1}a_{2j_2}\cdots a_{kj_k}\, e_{j_1}\wedge e_{j_2}\wedge\cdots\wedge e_{j_k}$$

$$= \sum_{1\le j_1<j_2<\cdots<j_k} \underbrace{\sum_{\sigma\in S_k} \mathrm{sgn}(\sigma)\, a_{1j_{\sigma(1)}}\cdots a_{kj_{\sigma(k)}}}_{\text{行列式的表达式}}\, e_{j_1}\wedge e_{j_2}\wedge\cdots\wedge e_{j_k}$$

?

$\sum_{1\le j_1\ne j_2\ne\cdots\ne j_k\le n} a_{1j_1}a_{2j_2}\cdots a_{kj_k}\, e_{j_1}\wedge e_{j_2}\wedge\cdots\wedge e_{j_k}$ 是 $j_1,\dots,j_k$ 的全部可能排列

也即取定从小到大排的一组数

$$\sum_{1\le j_1<j_2<\cdots<j_k\le n} a_{1j_{\sigma(1)}}a_{2j_{\sigma(2)}}\cdots a_{kj_{\sigma(k)}}\, \underbrace{e_{j_{\sigma(1)}}\wedge e_{j_{\sigma(2)}}\wedge\cdots\wedge e_{j_{\sigma(k)}}}_{\text{用}\mathrm{sgn}(\sigma)\text{整理成按序排}}$$

$$= \sum_{1\le j_1<j_2<\cdots<j_k\le n} a_{1j_{\sigma(1)}}a_{2j_{\sigma(2)}}\cdots a_{kj_{\sigma(k)}} \sum_{\sigma\in S_k} \mathrm{sgn}(\sigma)\, e_{j_1}\wedge e_{j_2}\wedge\cdots\wedge e_{j_k}$$

Slogan: Wedge 基展开系数是原向量矩阵 minors.

- Plücker embedding $G(k,n) \longrightarrow \mathbb{P}^{\binom{n}{k}-1}$

Lemma 1, 2 对于构造 Plücker embedding 非常重要.

Δ Lemma 1: $v_1,\dots,v_k\in k^n$ for some $k\le n$. Then

$v_1\wedge v_2\wedge\cdots\wedge v_k = 0$ iff $v_1,\dots,v_k$ are Lin. dep.

pf: Idea: use minors to compute wedge.

Let $v_i = \sum_{j=1}^{n} a_{ij}e_j$, $i\in[1,k]$

$$v_1\wedge\cdots\wedge v_k = 0 \Leftrightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix} =: A \text{ 的所有 } k \text{ 阶 minors 为 } 0 \Leftrightarrow \operatorname{rank} A < k$$

$$\Leftrightarrow v_1\cdots v_k \text{ Lin. dep.}$$

Δ Lemma 2: $v_1,\dots,v_k\in k^n$ and $w_1,\dots,w_k\in k^n$ both be Lin. ind.

Then $\boxed{v_1\wedge\cdots\wedge v_k \text{ and } w_1\wedge\cdots\wedge w_k \text{ Lin. ind. in } \wedge^k k^n \text{ (差常数倍)}} \Leftrightarrow \boxed{\mathrm{Lin}(v_1,\dots,v_k) = \mathrm{Lin}(w_1,\dots,w_k)}$

pf:

$\Rightarrow$ Assume $w_1\wedge\cdots\wedge w_k = c\, v_1\wedge\cdots\wedge v_k$

正推不容易，我们去找等价条件.

$v_i \in \mathrm{Lin}(w_1,\dots,w_k) \Leftrightarrow v_i, w_1,\dots,w_k$ Lin. dep. $\Leftrightarrow v_i\wedge w_1\wedge\cdots\wedge w_k = 0$

$\Leftrightarrow \underline{v_i\wedge v_1\wedge\cdots\wedge v_k\cdot c = 0}$ 出现两个v_i，恒成立. $\Rightarrow \mathrm{Lin}(v_1,\dots,v_k)\subseteq \mathrm{Lin}(w_1,\dots,w_k)$

同理有另一边包含.

$\Leftarrow$ Assume $\mathrm{Lin}(v_1,\dots,v_k) = \mathrm{Lin}(w_1,\dots,w_k)$

Then
$$\begin{aligned} v_1 &= c_{11}w_1 + \cdots + c_{1k}w_k \\ &\ \ \vdots \\ v_k &= c_{k1}w_1 + \cdots + c_{kk}w_k \end{aligned} \quad\rightsquigarrow\quad v_1\wedge\cdots\wedge v_k = \begin{vmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & & \vdots \\ c_{k1} & \cdots & c_{kk}\end{vmatrix} w_1\wedge\cdots\wedge w_k$$

矩阵可逆, $\det\neq 0$ 故二者 Lin. dep.

* Slogan: Wedge 刻画 ①是否 Lin. ind. (wedge $\neq 0$)
②是否张成相同空间 (wedge 差个非0系数)

Δ Plücker embedding: $f: G(k,n) \longrightarrow \mathbb{P}^{\binom{n}{k}-1}$

$\mathrm{Lin}(v_1,\dots,v_k) \longmapsto v_1\wedge\cdots\wedge v_k \in \wedge^k K^n \cong K^{\binom{n}{k}}$

$\mathrm{Lin}(v_1,\dots,v_k)\in G(k,n)$ 在 $\mathbb{P}^{\binom{n}{k}-1}$ 中的 plücker coordinate 是 $k\times k$-minors of matrix whose rows are $v_1, v_2,\dots,v_k$ 用基展开的系数.
(maximal minors)

* well-defined: If $v_1\wedge\cdots\wedge v_k = 0 \Rightarrow v_1,\dots,v_k$ Lin. dep. $\Rightarrow \dim \mathrm{Lin}(v_1,\dots,v_k) < k$
$\Rightarrow \mathrm{Lin}(v_1,\dots,v_k)\notin G(k,n)$

If $\mathrm{Lin}(w_1,\dots,w_k) = \mathrm{Lin}(v_1,\dots,v_k) \Rightarrow w_1\wedge\cdots\wedge w_k = \lambda v_1\wedge\cdots\wedge v_k,\ \lambda\in K^*$

$\Rightarrow w_1\wedge\cdots\wedge w_k = v_1\wedge\cdots\wedge v_k$ in $\mathbb{P}^{\binom{n}{k}-1} = K^{\binom{n}{k}}/K^*$

* f inj: $v_1\wedge\cdots\wedge v_k = w_1\wedge\cdots\wedge w_k \Rightarrow \mathrm{Lin}(v_1\wedge\cdots\wedge v_k) = \mathrm{Lin}(w_1\wedge\cdots\wedge w_k)$

只要再证 $\mathrm{Im} f$ closed in $\mathbb{P}^{\binom{n}{k}-1}$ 即可证明 $G(k,n)$ 是 proj. var.

• $G(k,n)$ is a proj. var.

* Examples convinced this fact:

$G(1,n) \longrightarrow \mathbb{P}^{\binom{n}{1}-1} = \mathbb{P}^{n-1}$

$\mathrm{Lin}(a_1e_1+\cdots+a_ne_n) \longmapsto (a_1:\cdots:a_n)$ $\quad\rightsquigarrow G(1,n) = \mathbb{P}^{n-1}$

* Idea of showing $\mathrm{Im} f$ closed: $\mathrm{Im} f =$ "pure wedges" $\leftarrow$ find equation to describe pure wedge

Δ Lemma: For a fixed non-zero $\omega\in\wedge^k K^n$ with $k<n$, consider k-linear map $g: K^n \longrightarrow \wedge^{k+1}K^n,\ v\longmapsto v\wedge\omega$

Then $\mathrm{rk} f \geq n-k$. The equation holds iff $\underline{\omega = v_1\wedge\cdots\wedge v_k}$ for some $v_1,\dots,v_k\in K^n$

ω is "pure" or in image of plücker embedding

pf: $\boxed{\mathrm{rk}\, g + \dim\ker g = n}$

Assume $v_1,\dots,v_r$ are basis of $\ker g$. Extend them to basis of K^n: $v_1, v_2,\dots,v_r, v_{r+1},\dots,v_n$. $\omega\in\wedge^k K^n$, so we assume

$$\omega = \sum_{1\leq j_1<\cdots<j_k\leq n} a_{j_1\cdots j_k} v_{j_1}\wedge\cdots\wedge v_{j_k}.$$ 由于 $\omega\neq 0$, $\exists\, a_{j_1\cdots j_k}\neq 0$.

由于 $v_i\in\ker g$, we have $v_i\wedge\omega = 0 = \sum_{(j_1,\dots,j_k)} a_{j_1\cdots j_k} v_i\wedge v_{j_1}\wedge\cdots\wedge v_{j_k}$

When $(j_1,\cdots,j_k)$ ranges, there are two cases (1) $i\in(j_1,\cdots,j_k)$

(2) $i\notin(j_1,\cdots,j_k)$

(1) If $i\in\{j_1,\cdots,j_k\}$, then $v_i\wedge v_{j_1}\wedge\cdots\wedge v_{j_k}=0$. $a_{j_1\cdots j_k}$ can be nonzero

(2) $v_i\wedge w$ 只留下非0项:

$$v_i\wedge\omega=0=\sum_{i\notin(j_1,\cdots,j_k)} a_{j_1\cdots j_k}\underbrace{v_i\wedge v_{j_1}\wedge\cdots\wedge v_{j_k}}_{\text{basis in }\wedge^{k+1}k^n}$$ so they are all Lin. ind.!

Hence $a_{j_1\cdots j_k}=0$ for $\forall(j_1,\cdots,j_k)\not\ni i$

以上分析对 $r\in[1,r]$ 都成立. 故

Conclusion: 只有满足 $\{1,\cdots,r\}\subseteq\{j_1,\cdots,j_k\}$ 的 $(j_1,\cdots,j_k)$ 才有

$a_{j_1\cdots j_r}\neq0$. 已知 $\exists a_{j_1\cdots j_r}\neq0$, 故 $\{j_1,\cdots,j_k\}\supseteq\{1,\cdots,r\}$

于是 $r=\dim\ker g\leq k$. 即 $\mathrm{rk}\,g=n-\dim\ker g=n-r\geq n-k$.

若等号成立, $r=\dim\ker g=k.\Rightarrow\exists\{j_1,\cdots,j_k\}=\{1,\cdots,r\}$.

则 $\omega=a_{1\cdots k}v_1\wedge\cdots\wedge v_k$ pure.

key pt: ω 只留下 $\{j_1\cdots j_k\}\supseteq\{1,\cdots,r\}$ terms.

Exp: $k=2$, $n=4$

(1) $\omega=e_1\wedge e_2$.

$$g(a_1e_1+\cdots+a_4e_4)=(a_1e_1+\cdots+a_4e_4)\wedge e_1\wedge e_2$$
$$=a_3\underline{e_1\wedge e_2}\wedge e_3+a_4\underline{e_1\wedge e_2}\wedge e_4$$

只留下含 e_1,e_2 terms. (这时 $\ker g=\mathrm{Lin}(e_1,e_2)$)

$\mathrm{rk}\,g=2=n-k$

(2) $\omega=e_1\wedge e_2+e_3\wedge e_4$

$$g(a_1e_1+\cdots+a_4e_4)=(a_1e_1+\cdots+a_4e_4)\wedge(e_1\wedge e_2+e_3\wedge e_4)$$
$$=a_1e_1\wedge e_3\wedge e_4+a_2e_2\wedge e_3\wedge e_4+a_3e_1\wedge e_2\wedge e_3+a_4e_1\wedge e_2\wedge e_4$$

$\mathrm{rk}\,f=4>n-k$

Δ The image of plücker embedding is closed in $\mathbb{P}^{\binom{n}{k}-1}$

pf: $0\neq\omega\in\wedge^kk^n\cong k^{\binom{n}{k}}$. $\underbrace{[\omega]}_{\text{modulo }k^*}\in\mathbb{P}^{\binom{n}{k}-1}$.

$[\omega]\in\mathrm{Im}f\Leftrightarrow\omega$ is pure $\Leftrightarrow g:k^n\to\wedge^{k+1}k^n$ $v\mapsto v\wedge\omega$ has $\mathrm{rk}\,g=n-k$

$\Leftrightarrow\underbrace{\text{all }(n-k+1)\text{-minors}}_{\text{They are all homo polys}}$ of matrix g are zeros, since $\mathrm{rk}\,g\geq n-k$

So those ω forms a closed set, i.e., plücker embedding is closed.

Exp: $G(2,4)=V_p(?)$ in $\mathbb{P}^{\binom{4}{2}-1}=\mathbb{P}^5$

$$g:k^4\longrightarrow\wedge^3k^4$$
$$v\longmapsto v\wedge\omega$$

$\omega=a_{12}e_1\wedge e_2+a_{13}e_1\wedge e_3+a_{14}e_1\wedge e_4+a_{23}e_2\wedge e_3+a_{24}e_2\wedge e_4+a_{34}e_3\wedge e_4$

Coordinates of $G(2,4)$ in $\mathbb{P}^5$

☆ key: 把 $\mathbb{P}^5$ 视作 $\wedge^2k^4/k^*$

$$e_1\longmapsto e_1\wedge\omega=a_{23}e_1\wedge e_2\wedge e_3+a_{24}e_1\wedge e_2\wedge e_4+a_{34}e_1\wedge e_3\wedge e_4$$

$v \longmapsto v \wedge w$

$e_1 \longmapsto a_{23} e_1\wedge e_2\wedge e_3 + a_{24} e_1\wedge e_2\wedge e_4 + a_{34} e_1\wedge e_3\wedge e_4$

$e_2 \longmapsto -a_{13} e_1\wedge e_2\wedge e_3 - a_{14} e_1\wedge e_2\wedge e_4 + a_{34} e_1\wedge e_3\wedge e_4$

$e_3 \longmapsto a_{12} e_1\wedge e_2\wedge e_3 - a_{14} e_1\wedge e_3\wedge e_4 - a_{24} e_2\wedge e_3\wedge e_4$

66 $e_4 \longmapsto a_{12} e_1\wedge e_2\wedge e_4 + a_{13} e_1\wedge e_3\wedge e_4 + a_{23} e_2\wedge e_3\wedge e_4$ Andreas Gathmann

$$\begin{array}{c|cccc} & e_1\wedge e_2\wedge e_3 & e_1\wedge e_2\wedge e_4 & e_1\wedge e_3\wedge e_4 & e_2\wedge e_3\wedge e_4 \\ \hline e_1 & a_{23} & a_{24} & a_{34} & 0 \\ e_2 & -a_{13} & -a_{14} & 0 & a_{34} \\ e_3 & a_{12} & 0 & -a_{14} & -a_{24} \\ e_4 & 0 & a_{12} & a_{13} & a_{23} \end{array}$$

这个矩阵所有$\underset{\overset{\shortparallel}{3}}{(n-k+1)}$-minors 为0

共有16条 equs.

故 $G(2,4)=V_p(\text{all } 3\text{-minors})$

事实上这16条 equ 是冗余的，只要一条方程就可以. 这带来一个麻烦：用这种办法并不能清楚地看到 $\dim G(k,n)=?$

- Compute $\dim G(k,n)=k(n-k)$

Idea: Note that we have this prop:

If $\{U_i \mid i\in I\}$ is an open cover of X then $\dim X=\sup\{\dim U_i \mid i\in I\}$

我们将 $G(k,n)$ 用 open aff. patch cover，就可以计算 $\dim G(k,n)$.

$G(k,n)\subseteq \mathbb{P}^{\binom{n}{k}-1}=\bigcup_i U_i \leftarrow$ has aff. cover.

so $G(k,n)=\bigcup_i U_i\cap G(k,n)$ also an aff. cover of $G(k,n)$.

Let's look more carefully on these aff. patch.

$v_1\wedge\cdots\wedge v_k\in f(G(k,n))\subseteq \mathbb{P}^{\binom{n}{k}-1}$ 的 plücker coord 是

$(a_{12\cdots k} : a_{12\cdots k-1\,k+1} : \cdots)$

$a_{12\cdots k-1\,k+1}$：第 $1,2,\cdots,k-1,k+1$ 列组成的子矩阵的行列式

$a_{12\cdots k}$：第 $1,2,\cdots,k$ 列组成的子矩阵行列式

$$\begin{bmatrix} v_1\\ \vdots \\ v_k\end{bmatrix}=\begin{bmatrix} a_{11} & \cdots & a_{1n}\\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn}\end{bmatrix}\begin{bmatrix} e_1\\ \vdots\\ e_n\end{bmatrix}$$

用 $[A\ B]$ 表示 $\begin{bmatrix} v_1\\ \vdots \\ v_k\end{bmatrix}$

因此 $f(G(k,n))$ 的 aff. patch 是某个 k-minor 非0.

规定 U_0 是第1至k列组成的子矩阵的行列式. 则 $f(G(k,n))$ 的 patch 是 $[A\ \ B]$.

with A invertible. 转到 $G(k,n)$ 中，

$G(k,n)$ 的 aff. patch U_0 是 $\mathrm{Lin}\begin{bmatrix} v_1\\ \vdots \\ v_k\end{bmatrix}=$ row span of $[A\ \ B]$

$=$ row span of $k[\underset{k}{I}\ \ \overset{n-k}{C}]$ where $C=A^{-1}B\in M_{(n-k)\times k}$

Hence $U_0\cong \mathbb{A}^{k(n-k)}$

$G(k,n)$ 的每一片 patch 都同构于 $\mathbb{A}^{k(n-k)}$，因此 $\dim G(k,n)=k(n-k)$.

*Moreover $G(k,n)$ is irr.

Fact: $\{U_i\}_{i\in I}$ be an open cover of topo space X and $U_i\cap U_j\neq\emptyset$ for all i,j. Then if U_i is irr for all $i\in I$, we have X is irr.

Since $\mathbb{A}^{k(n-k)}$ is irr(open set in irr $\mathbb{P}^{k(n-k)}$) so $G(k,n)$ is irr.

* $G(k,n)=\{\text{full rank } k\times n \text{ matrix}\}/GL(k)$，表示 row transformation.

$\mathrm{Lin}\begin{pmatrix} v_1\\ \vdots \\ v_k\end{pmatrix}=\text{row span}[\underset{\underset{\text{rank } k}{\uparrow}}{A}\ \ B] \xlongequal{\text{行变换不改变 row span}}$ row span 阶梯型矩阵

Exp: $G(1,2)=\{\text{full rank } 1\times 2 \text{ matrix}\}/GL(k)=\{\overset{\overset{\shortparallel}{\mathbb{A}^1}}{(1\ *)},\ \overset{\infty \text{ in } \mathbb{P}^1}{(0\ 1)} \mid *\in K\}$

$U_0=\{(I\ C)\mid C\in M_{(n-k)\times k}\}$ is an aff. space patch of $G(k,n)$

$\left\{\begin{pmatrix} 1 & * & 0 & 0 & 0 & x\\ 0 & x & 1 & 0 & 0 & x\\ 0 & x & 0 & 1 & 0 & x\\ 0 & x & 0 & 0 & 1 & x\end{pmatrix}\right\}$ is also an aff. space of $\overset{2\times 4}{G(4,6)}$

*注意 $G(k,n)$ 不只含 $\{(I\ C)\}$，这只是一个 patch！

- For all $0 \le k < n$, we have $G(k,n) \cong G(n-k,n)$

Idea: Construct iso.

pf: $\varphi|_{U_0}: G(k,n) \longrightarrow G(n-k,n)$

$L=(I_k\ C) \longmapsto L^{\perp}$ (L的正交补)

bijection is obvious: $(L^{\perp})^{\perp}=L$. 下证 φ是 mor (同理可证 φ^{-1}也是mor, 只要把k换成n-k). 需要把 $L^{\perp}$ 的形式算出来.

经过硬凑, 发现 $(I_k\ C)\begin{pmatrix}-C\\ I_{n-k}\end{pmatrix}=0$. 故 $L^{\perp}=(-C^T\ \ I_{n-k})$

φ可以看作 $\mathbb{A}^{n(n-k)} \longrightarrow \mathbb{A}^{k(n-k)}$ 的map

$C \longmapsto -C^T$

$-C^T$的每一个分量是 C_{ij} 的 deg 1 poly, 因此是 mor.

每一片patch上 iso 粘出整体iso.

- Birational maps: Almost the same (contain iso open sets though var are not iso)

* Exps for noniso vars containing iso open sets (examples for birational)

(1) $\mathbb{P}^n \times \mathbb{P}^m \ncong \mathbb{P}^{n+m}$. $\mathbb{P}^n\times\mathbb{P}^m \supseteq \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m} \subseteq \mathbb{P}^{n+m}$

(2) $G(k,n) \supseteq \mathbb{A}^{k(n-k)} \subseteq \mathbb{A}^{k(n-k)}$

(3)

$V(x_1^2 - x_2^3)$ $\ncong$ $\mathbb{A}^1$

把singular pt 挖掉 iso to $\mathbb{A}^1\setminus 0$ $\quad \cup \quad \mathbb{A}^1\setminus 0 \quad \cup$

- Rational maps

非常有用,开集相交非空

△ Def: X, Y: irr var. A rational map f from X to Y, written $f: X \dashrightarrow Y$, is a mor $f: U \longrightarrow Y$. We say $f_1, f_2: X \dashrightarrow Y$ defined on U_1 resp. U_2 are the same if $f_1 = f_2$ on a non-empty open subset of $U_1 \cap U_2$.

* $f: X \dashrightarrow Y$ defined on U 也记作 $f: U \longrightarrow Y$.

* Exp: $f_1: \mathbb{A}^1\setminus\{0\} \longrightarrow \mathbb{A}^1,\ x \longmapsto \frac{1}{x}$

  $f_2: \mathbb{A}^1\setminus\{0,1\} \longrightarrow \mathbb{A}^1,\ x \longmapsto \frac{1}{x}$

  $f_1 = f_2$ as rational maps.

* 这是等价关系. 反身性、对称性显然. Check 传递性:

$f_1: U_1 \longrightarrow Y$ agree with $f_2: U_2 \longrightarrow Y$ on an open set $U_{12} \overset{open}{\subseteq} U_1 \overset{open}{\cap} U_2$

$f_2: U_2 \longrightarrow Y$ agree with $f_3: U_3 \longrightarrow Y$ on an open set $U_{23} \overset{open}{\subseteq} U_2 \overset{open}{\cap} U_3$

$U_{12} \cap U_{23} \overset{open}{\subseteq} U_1 \cap U_2 \cap U_3 \subseteq U_1 \cap U_3$. 只要证 $U_{12} \cap U_{23} \neq \emptyset$.

X irr, $U_1 \cap U_3 \overset{open}{\subseteq} X \Rightarrow U_1 \cap U_3$ irr. $U_{12} \overset{open}{\subseteq} U_1 \cap U_3$, $U_{23} \overset{open}{\subseteq} U_1 \cap U_3$

$\xRightarrow{U_1 \cap U_3 \text{ irr}} U_{12} \cap U_{23} \neq \emptyset$.

* $f_1: U_1 \longrightarrow Y$, $f_2: U_2 \longrightarrow Y$ are the same $\Rightarrow f_1|_{U_1\cap U_2} = f_2|_{U_1 \cap U_2}$.

pf: Let $W = \{x \in U_1 \cap U_2 \mid f_1(x) = f_2(x)\}$

$= i^{-1}(\Delta_Y)$ is closed in $U_1 \cap U_2$, where $i: U_1 \cap U_2 \xrightarrow{(f_1, f_2)} Y \times Y$

$x \longmapsto (f_1(x), f_2(x))$

f_1 and f_2 are the same $\Rightarrow \exists$ open set $U \subseteq U_1 \cap U_2$, $f_1|_U = f_2|_U$

Obviosly $\underbrace{U \subseteq W}_{\text{open}} \overset{\text{closed}}{\subseteq} U_1 \cap U_2$.

Since $W \overset{\text{closed}}{\subseteq} U_1 \cap U_2$, $\bar{U} \subseteq W$. However U is an open set in $U_1 \cap U_2$, so $\bar{U} = U_1 \cap U_2$. Then $U_1 \cap U_2 = \bar{U} \subseteq W$ meaning that $U_1 \cap U_2 \subseteq W$

So $W = U_1 \cap U_2$, i.e., $f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$.

- Define birational

* Intuitively, X and Y are birational if

$\exists f: X \dashrightarrow Y$, $g: Y \dashrightarrow X$ with $f \circ g = id_Y$, $g \circ f = id_X$

But it may run to problems in some case:

$f: X \dashrightarrow Y$ with $f(x)$ closed in Y and

$g: Y \dashrightarrow X$ is defined on $Y \setminus f(x)$. $\rightsquigarrow g \circ f$ is meaningless!

Hence we need concept "dorminant" just ensure composition is well-def.

△ Def: (Dorminant) $f: X \dashrightarrow Y$ is called dorminant

if (TFAE)
- $f(x)$ dense in Y for all representations
- $f(x)$ contains a nonempty open subset of Y for all rep.

$$U \xrightarrow{f} Y \xrightarrow{g} Z$$

- f dorminant, so $\exists V \subseteq f(U)$, $V \overset{\text{open}}{\subseteq} Y$
- assume g defined on $W \overset{\text{open}}{\subseteq} Y$

$W \cap V \neq \emptyset$, since Y irr

△ A rational map $f: X \dashrightarrow Y$ is called birational if f is dorminat and $\exists$ dorminant $g: Y \dashrightarrow X$ with $g \circ f = id_X$ and $f \circ g = id_Y$

* We say X and Y are birational if $\exists$ birational $f: X \dashrightarrow Y$

* If X is birational to a proj. space $\mathbb{P}^n$ we call X is rational.

△ two irr var are birational $\Leftrightarrow$ they contain iso. non-empty open sets.

$\Rightarrow$ $f: X \dashrightarrow Y$ birational defined on U with inverse $g: Y \to X$ defined on V, then $U \cap f^{-1}(V)$ iso to $V \cap g^{-1}(U)$ by f and g.

$\Leftarrow$ $X \overset{\text{open}}{\supseteq} U \overset{f}{\simeq} V \overset{\text{open}}{\subseteq} Y$ $f: X \dashrightarrow Y$ defined on U is birational.

△ Fact: Birational irr var have the same dimension.

△ Example of birational maps —— Rational functions and function fields

* A rational function on X is a rational function $\varphi: X \dashrightarrow \mathbb{A}^1$ given by $\varphi \in \mathcal{O}_X(U)$ on some $U \overset{\text{open}}{\subseteq} X$. The set of all rational functions on X will be denoted by $\underline{K(X)}$. a field $\varphi_1 + \varphi_2$, $\varphi_1 \cdot \varphi_2$ defined on $U_1 \cap U_2$

$-\varphi_1$ on U_1 and φ_1^{-1} on $U_1 \setminus V(\varphi_1)$

* $\emptyset \neq U \overset{\text{open}}{\subseteq} \underset{\text{irr}}{X}$, then $K(U) \cong K(X)$ iso 是 obvious.

我们总可取 aff. open $U \subseteq X$，于是问题变成 $K(\text{aff.})$ 是什么?

U aff. open in irr $X \Rightarrow U$ is irr $\Rightarrow A(U)$ is a domain

Then $K(U)$ 是 $A(U)$ 的分式域. (Reason$^{\text{nonstrictly}}$: U 可以写成 $D(f_i)$ 之并.

$K(U) \cong K(D(f_i))$. $D(f_i)$ 上 function 永远可以写成 $\frac{g}{f}$)

- Blow up : a general procedure 把 irr var 变成与原来 birational 的新 irr. var.

△ Def: Let $X \subseteq \mathbb{A}^n$ an aff. var. For some given $f_1,\cdots,f_r \in A(X)$

set $U = X \setminus V(f_1,\cdots,f_r)$, there is a well-defined mor

$$f: U \longrightarrow \mathbb{P}^{r-1},\quad x \mapsto (f_1(x): f_2(x): \cdots : f_r(x))$$

Consider $\Gamma_f = \{(x,f(x)) \mid x \in U\} \subseteq U \times \mathbb{P}^{r-1}$ which is closed in $U \times \mathbb{P}^{r-1}$

The closure of Γ_f in $X \times \mathbb{P}^{r-1}$ is called blow up of X at $f_1,\cdots,f_r$

denoted by $\tilde{X}$. There is projection $\pi: \tilde{X} \to X$

△ (1) $\Gamma_f = \tilde{X} \cap U \times \mathbb{P}^{r-1}$

(2) $\pi: \tilde{X} = \Gamma_f \cup \tilde{X} \setminus \Gamma_f \longrightarrow X = U \cup X \setminus U$

$\Gamma_f \xrightarrow{\text{iso}} U$, $\quad \tilde{X} \setminus \Gamma_f \xrightarrow{\text{not iso}} X \setminus U$

☆ $\tilde{X} = \overline{U} = \overline{\Gamma_f}$

* $\Gamma_f \cong U$ be a dense ($\bar{U} = X$) open subset.

如我们所愿，我们造了一个 $\tilde{X}$ birational to X (iso on an open subset)

Note that : we sometimes identify Γ_f as U (view $U \subseteq \tilde{X}$)

△ Blow up of subvar. Let Y be a closed var of X.

$U \cap Y = Y \setminus V(f_1,\cdots,f_r)$ $\quad \tilde{Y}$ is the closure of $\Gamma_{f|_{U\cap Y}}$ in $Y \times \mathbb{P}^{r-1}$,

where $f|_{U\cap Y}: U \cap Y \to \mathbb{P}^{r-1}$

(1) $\tilde{Y}$ is closed in $\tilde{X}$. It is called strict transform of Y in $\tilde{X}$.

$$\tilde{Y} \overset{\text{closed}}{\subseteq} Y \times \mathbb{P}^{r-1} \overset{\text{closed}}{\subseteq} X \times \mathbb{P}^{r-1} \Rightarrow \tilde{Y} \text{ closed in } X \times \mathbb{P}^{r-1}$$

$$\Rightarrow \tilde{Y} = \underbrace{\tilde{Y}}_{\text{closed in } X \times \mathbb{P}^{r-1}} \cap \tilde{X} \overset{\text{closed}}{\subseteq} \tilde{X}$$

Moreover

(2) $\tilde{Y}$ is closure of $Y \cap U$ in $\tilde{X}$.

$$\overline{Y \cap U}_{\tilde{X}} \underset{\tilde{Y} \text{ closed in } \tilde{X}}{=} \overline{Y \cap U}_{\tilde{Y}} = \tilde{Y}$$

* If $X = \bigcup_i X_i$ ← irr decomposition, $\tilde{X} = \bigcup_i \tilde{X}_i$

$\tilde{X}_i$ is closure of $X_i \cap U$ in $\tilde{X}$. 取 closure 与取有限并对易.

△ Exp: Compute by taking closure

$r = 1$. $\quad U = X \setminus V(f_1)$. $\quad \tilde{X} \subseteq X \times \mathbb{P}^0 \cong X$.

so $\tilde{X}$ is closure of U in X (under iso)

(1) $f_1 \neq 0$. $\quad \overline{U} = X = \tilde{X}$.

(2) $f_1 = 0$ $\quad U = X \setminus V(0) = \phi \quad \tilde{X} = \overline{U} = \phi$.

- Computation — Description not refer to taking closures.

△ Lemma: The blow up $\tilde{X}$ of an aff. var X at $f_1, \dots, f_r \in A(X)$ satisfies $\tilde{X} \subseteq \{(x,y) \in X \times \mathbb{P}^{r-1} \mid y_i f_j(x) = y_j f_i(x) \text{ for all } i,j=1 \dots r\}$
$=: Z$

pf: $\Gamma_f = \{(x, [f_1(x), \dots, f_r(x)])$ 显然满足方程 $(y_i = f_i(x))$

Z closed in $X \times \mathbb{P}^{r-1}$ so $\tilde{X} = \overline{\Gamma_f} \subseteq Z$.

△ Exp: Blow up of $\mathbb{A}^n$ at the coord. funs.

* Lemma 1: $X \overset{\text{closed}}{\subseteq} Y$, X, Y irr, $\dim X = \dim Y \Rightarrow X = Y$

pf If $X \subsetneq Y$, then by def of dim, we have

$\cdots \subsetneq X_n \subsetneq Y \quad \dim Y \geq \dim X + 1 \rightarrow\leftarrow$

* Lemma 2: If X is irr, then $\tilde{X}$ is irr.

$X = \overline{U}^X$, X irr so U is irr. Then $\overline{U}^{\tilde{X}} = \tilde{X}$ irr. (U irr $\Leftrightarrow \overline{U}$ irr)

* $X = \mathbb{A}^n$, $f_1 = x_1$, $f_2 = x_2, \dots, f_n = x_n$.

$\tilde{\mathbb{A}}^n \subseteq \{(x, [y_1, \dots, y_n]) \in X \times \mathbb{P}^{n-1} \mid y_j x_i = x_j y_i, \forall i,j\} =: Z$

Z has an aff. open cover: $Z \cap \{y_i \neq 0\}$.

We can pick $y_i = 1$, then $Z \cap \{y_i \neq 0\} = \{(x, [y_1, \dots, \overset{i\text{-th}}{1}, \dots y_n]) \in X \times \mathbb{P}^{n-1} \mid$
$y_j x_i = x_j, \forall j\}$

从equ中梳理真正的变量

$$U_i := Z \cap \{y_i \neq 0\} = \{((y_1 x_i, y_2 x_i, \dots, y_n x_i), [y_1, \dots, \overset{y_i}{\overset{\shortparallel}{1}}, \dots, y_n])\}$$

$$\cong \{(x_i, y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)\} \cong \mathbb{A}^n$$

$\begin{cases} U_i \cap U_j \neq \emptyset \\ Z = \bigcup_i U_i \text{ is an open cover} \end{cases} \Rightarrow \dim Z = \sup_i \dim U_i = n$

$\begin{cases} U_i \cap U_j \neq \emptyset \\ Z = \bigcup_i U_i \text{ is an open cover} \\ U_i \cong \mathbb{A}^n \text{ irr} \end{cases} \Rightarrow Z \text{ is irr}$

$\mathbb{A}^n$ birational to $\tilde{\mathbb{A}}^n$ so $\dim \tilde{\mathbb{A}}^n = \dim \mathbb{A}^n = n$.

$\mathbb{A}^n$ is irr so $\tilde{\mathbb{A}}^n$ is irr. Then $\begin{cases} \tilde{\mathbb{A}}^n \overset{\text{closed}}{\subseteq} Z \\ \tilde{\mathbb{A}}^n, Z \text{ irr} \\ \dim \tilde{\mathbb{A}}^n = n = \dim Z \end{cases} \Rightarrow \tilde{\mathbb{A}}^n = Z$

Question: What's the picture of $\widetilde{\mathbb{A}}^2$?

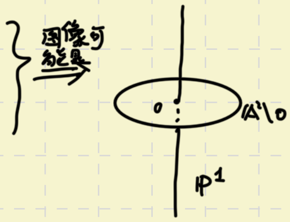
* Consider $\pi: \widetilde{\mathbb{A}}^2 \to \mathbb{A}^2$, where $\widetilde{\mathbb{A}}^2 = Z = \{(x,y) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid y_2x_1 = y_1x_2\}$

i) $\pi^{-1}(0) = \{(0,y) \in \mathbb{A}^n \times \mathbb{P}^{n-1} \mid y_i 0 = y_j 0\}$

$= \{(0,y) \in \mathbb{A}^n \times \mathbb{P}^{n-1}\} \cong \mathbb{P}^{n-1}$.

ii) $\widetilde{\mathbb{A}}^2 \setminus \pi^{-1}(0) \cong \mathbb{A}^2 \setminus 0$

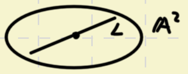
图像可能是



But it's wrong: $\mathbb{A}^2$ irr $\Rightarrow$ $\widetilde{\mathbb{A}}^2$ irr, BUT in the right picture it's reducible.

Correct picture: To obtain correct picture,

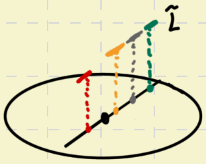
1) Let's consider line $L \subseteq \mathbb{A}^2$



2) Compute strict transformation of L in $\widetilde{\mathbb{A}}^2$

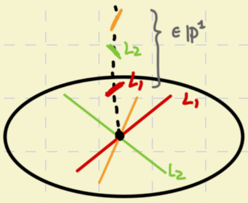
设L的斜率是k. $\forall$ point on L can be represented by (x_1, kx_1)

So the image is $((x_1, kx_1), [x_1 : kx_1]) \in \mathbb{A}^2\setminus 0 \times \mathbb{P}^1$



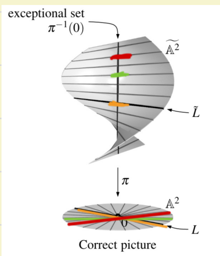
合理猜测 $\tilde{L} \cap \pi^{-1}(0) = (0, [1:k])$

3) 不同 line L 与 $\pi^{-1}(0)$ 的交.



$\rightsquigarrow$ Exceptional set parametrize the direction (L_1, L_2, L_3) in $\widetilde{\mathbb{A}}^2$

$\widetilde{\mathbb{A}}^2$ 示意图:



Correct picture

☆ 原来相交的线在 Blow up 中都分开了

☆ 这是 Mobiüs 带.

• prop: The blow up of an aff. var X at $f_1, \dots, f_r \in A(X)$ depends only on the ideal $\langle f_1, \dots, f_r \rangle \trianglelefteq A(X)$, i.e., if $\langle f_1, \dots, f_r \rangle = \langle f_1', \dots, f_s' \rangle$, and $\pi: \tilde{X} \to X$ and $\pi': \tilde{X}' \to X$ are corresponding blow ups, there is an iso $F: \tilde{X} \to X$ with

$$\begin{tikzcd} \tilde{X} \arrow[rr, "F"] \arrow[dr, "\pi"'] & & \tilde{X}' \arrow[dl, "\pi'"] \\ & X \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

* Blow up 只与 ideal 有关, independent of 生成元选取.

pf: (Routine check) Construct F. Trick: 两套生成元必可互相表示.

Assume $f_i = \sum g_{il} f_l'$, $f_j' = \sum h_{jk} f_k$.

$F: \tilde{X} \longrightarrow \tilde{X}'$

$(x, [y_1, \dots, y_r]) \mapsto (x, [\cdots, \sum h_{jk} f_k, \cdots])$

Idea: 已知在 $X\backslash V(f_1,\cdots,f_r) \subseteq \tilde{X}$ 上的点

坐标形如 $(x, [f_1(x), f_2(x), \cdots, f_r(x)])$. 我们希望有

$$F|_{X\backslash V(f_1,\cdots,f_r)} : X\backslash V(f_1,\cdots,f_r) \longrightarrow \tilde{X}'$$

$$(x,[f_1(x),\dots,f_r(x)]) \mapsto (x,[f_1'(x),\dots,f_s'(x)])$$

$$= (x,[\cdots,\underbrace{\boxed{\textstyle\sum_k h_{jk}f_k}}_{k\text{-th component}},\cdots])$$

下面只需证 ① F是mor ② F是iso.

② F是iso非常容易, 只要构造

$$G:\tilde{X}'\to\tilde{X}$$

$$(x,[y_1,\dots,y_s])\mapsto(x,[\cdots,\textstyle\sum g_{il}f_l',\cdots])$$

如果证明F是mor, 则G是mor. 易知 $G=F^{-1}$, 故 F 是iso.

It suffices to show ① F is a mor.

1) F is well-defined $\begin{cases}(1.1) & [\cdots,\sum h_{jk}f_k,\cdots]\neq[0,0,\dots,0]\\(1.2) & F(\tilde{X})\subseteq\tilde{X}'\end{cases}$

(1.1)

Idea: $U=X\backslash V(f_1,\cdots,f_r)$. $\tilde{X}=\bar{U}$. 从取闭包的space入手.

已知 $\tilde{X}|_U=\{(x,[f_1(x),\cdots,f_r(x)])\}$. 显然 $\tilde{X}|_U\subseteq V((x,y): y_i=\sum_{j,k}g_{ij}h_{jk}y_k)$ closed

$\tilde{X}=\bar{U}$, so $\tilde{X}\subseteq V((x,y): y_i=\sum_{j,k}g_{ij}h_{jk}y_k)$. If $\exists\,(x,y)\in\tilde{X}$ with $\sum h_{jk}y_k=0\ \forall j$,

then $y_i=0\ \forall i$, i.e., $y=0$. It's impossible because $y\in\mathbb{P}^{r-1}$.

(1.2)

Idea: continue map 取逆才保闭集. w.t.s. $\tilde{X}\subseteq \underbrace{F^{-1}(\tilde{X}')}_{\text{closed}}$.

Idea: 当现闭集从取闭包的space U入手. $\tilde{X}=\bar{\Gamma}_f$. w.t.s. $\Gamma_f\subseteq F^{-1}(\tilde{X}')$

$\Gamma_f=\{(x,[f_1(x),\dots,f_r(x)])\}$表达式已知, check $F(\Gamma_f)\subseteq\tilde{X}'$更容易.

$$F(\Gamma_f)=\{(x,[\cdots,\textstyle\sum h_{jk}f_k(x),\cdots])\}=\{(x,[\cdots,f_j',\cdots])\}=\Gamma_{f'}\subseteq\tilde{X}'$$

So we complete the proof.

- Blow up 只由 ideal 决定 $\Longrightarrow$ 把对多项式作 Blow up 改成对多项式生成理想的 blow up.

1. $J\trianglelefteq A(X)$ be an ideal. We define blow up of X at J to be the blow up of X at any set of generators of J.
2. $Y\subseteq X$ be closed subvariety. We define blow up of X at Y to be the blow up of X at $I(Y)\trianglelefteq A(X)$. $\rightsquigarrow$ Unique up to isomorphism
3. X is an arbitrary variety. $Y\subseteq X$ be a closed subvariety.

   Question: How to define blow up of X at Y?

   Pick an affine open cover $\{U_i : i\in I\}$ of X. $\tilde{U}_i :=$ blow up of U_i at closed subvariety $U_i\cap Y$. 把 $\tilde{U}_i$ 拼起来得到 blow up of X at Y.

4. Special case: Blow up at a point a. 取 aff. open cover $\{U_i : i\in I\}$ of X.

   $\tilde{U}_i=$ blow up of U_i at $a=\begin{cases}\text{blow up} & a\in U_i\\ U_i & a\notin U_i\end{cases}$. 假如 U_0 cover a.

   Compute $\tilde{U}_0\Rightarrow$ Glue $X\backslash a$ to $\tilde{U}_0$ along common open set $U\backslash\{a\}$.

Slogan: Blow up是local操作. Blow up at a point 时总可取 aff. open set cover 住 this point. 因此总可认为 X is an aff. var.

5. Blow up of projective var.

   Slogan: Blow up of proj. var. is still proj. var.

Let X be a proj. var., $f_1, \cdots, f_r \in S(X)$ are homo with same degree.
Blow up of X at $f_1, \cdots, f_r$ is $\overline{\Gamma}$ in $X \times \mathbb{P}^{r-1}$ where Γ is:

$$\Gamma = \{(x, [f_1(x), f_2(x), \cdots, f_r(x)]) : x \in U := X \setminus V(f_1, \cdots, f_r)\} \subseteq U \times \mathbb{P}^{r-1}$$

$\overline{\Gamma}$ closed in proj var. $X \times \mathbb{P}^{r-1}$ so $\tilde{X} = \overline{\Gamma}$ is proj. var.

- Exceptional set of the blow up of a general variety X at point $a \in X$ parametrizes the tangent direction of X at a.

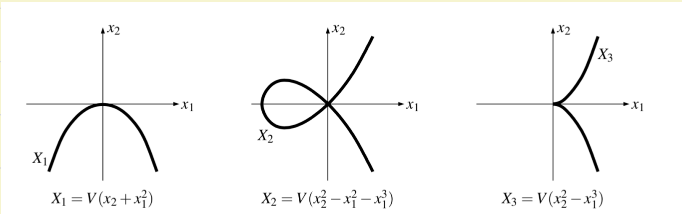
- Tangent cones. X: a variety. $\tilde{X}$:= blow up of X at a point $a \in X$.

$$\begin{cases} \pi^{-1}(a) \subseteq \{a\} \times \mathbb{P}^{r-1} \cong \mathbb{P}^{r-1}. \\ a \subseteq \text{closed} \Rightarrow \pi^{-1}(a) \text{ closed} \end{cases} \Rightarrow \pi^{-1}(a) \text{ is a projective variety.}$$

$\Downarrow$ $\pi^{-1}(a)$ 是 proj. var.

Cone over $\pi^{-1}(a)$ is called
Tangent Cone

Exp 1 Intuitive picture.



Consider strict tranformation of X_1, X_2, X_3 at origin:

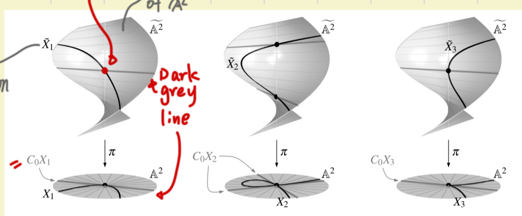
$(0, [1:k])$

blow up of $\mathbb{A}^2$

strict transformation of $\tilde{X}_1$

Dark grey line

$=$ line through $(1,k)$ in $\mathbb{A}^2$



point • in above: $\pi|_{\tilde{X}_1}^{-1}(0)$. By def of cone, it's line through point • in $\mathbb{A}^2$.
i.e., dark grey line in above. 正符合对切空间的理解.

Exp 2,3 convinced you:
- a 光滑, tangent cone 是 切空间
- a 不光滑, tangent cone 不是 切空间, 会包含 a 点附近信息.
- tangent cone 与光滑性研究有关.

Exp 2: (a 光滑) $X = V(x_2 - x_1^2) \subseteq \mathbb{A}^2$ $a = (0,0)$. compute $C_a(X)$.

Idea: $\tilde{X} \subseteq Z = \{(x, [y_1, y_2]) \in X \times \mathbb{P}^1 \mid y_1 x_2 = y_2 x_1\} \subseteq X \times \mathbb{P}^1$.

has two aff. patch $\leftarrow$ has two aff. patch $\longleftarrow$ has two aff. patch

$\tilde{X} = \Gamma_f$ consider Γ_f has two aff. patch

Trick: 用 mor 来说明 Z 的 aff. patch. $Z = U_1 \cup U_2$.

$\varphi: \mathbb{A}^2 \xrightarrow{\sim} U_1 \subseteq Z$, pick $y_1 = 1$
$(x_1, y_2) \mapsto ((x_1, y_2 x_1), [1 : y_2])$

$\psi: \mathbb{A}^2 \xrightarrow{\sim} U_2 \subseteq Z$, pick $y_2 = 1$
$(x_2, y_1) \mapsto ((y_1 x_2, x_2), [y_1 : 1])$

$$\Gamma_f \cap U_1 = \left\{ \left((x_1, y_2 x_1), [x_1 : y_2 x_1] \right) \middle| \ (x_1, y_2 x_1) \in \underset{X = V(x_2 - x_1^2)}{\underline{\underline{X}}} \setminus 0 \right\}$$

$$= \left\{ (x_1, y_2 x_1), [1 : y_2] \ \middle| \ \boxed{\begin{array}{l} (y_2 x_1) - x_1^2 = 0 \\ x_1 \neq 0 \end{array}} \right\}$$

$$= \left\{ (x_1, x_1 y_2), [x_1, x_1 y_2] \ \middle| \ \begin{array}{l} y_2 - x_1 = 0 \\ x_1 \neq 0 \end{array} \right\} \quad (x_1, y_2 x_1) \in X \setminus 0 \leftarrow \text{lies in } U_1 \cong \mathbb{A}^2$$

结合mor φ $\cong V(x_2 - x_1^2) \cap D(x_1) \subseteq \mathbb{A}^2 \cong U_1$

Closure of Γ_f in $Z = U_1 \cap U_2$ is $\tilde{X} = \bar{\Gamma}_f$.

Closure of $\Gamma_f \cap U_1$ in $Z \cap U_1 = U_1$ is $\tilde{X} \cap U_1$

$$\Gamma_f \cap U_1 = V(x_2 - x_1^2) \cap D(x_1) \overset{\text{open}}{\subseteq} \underset{\text{irr}}{V(x_2 - x_1^2)} \Rightarrow \Gamma_f \cap U_1 \text{ irr. in } V(x_2 - x_1^2) \text{ as sub var.}$$

If $A \subseteq B \overset{\text{closed}}{\subseteq} X$

then $\bar{A}^X = \bar{A}^B$

pf:

$\bar{A}^X \cap B = \bar{A}^B \Rightarrow \bar{A}^B \subseteq \bar{A}^X$

It suffices to show $\bar{A}^X \subseteq \bar{A}^B$.

$\bar{A}^B \overset{\text{closed}}{\subseteq} B \overset{\text{closed}}{\subseteq} X \Rightarrow A \subseteq \bar{A}^B \overset{\text{closed}}{\subseteq} X$

$\Rightarrow \bar{A}^X \subseteq \bar{A}^B$

$\Rightarrow \overline{\Gamma_f \cap U_1} = V(x_2 - x_1^2)$ as closure in $\underset{\text{closed}}{V(x_2 - x_1^2)}$

$\Rightarrow$ closure of $\Gamma_f \cap U_1$ in U_1

$$= \overline{\Gamma_f \cap U_1}^{V(x_2 - x_1^2)} = V(x_2 - x_1^2)$$

$$\Gamma_f \cap U_2 = \left\{ \left((y_1 x_2, x_2), [y_1 : 1] \right) \in X \setminus 0 \times \mathbb{P}^1 \right\}$$

$$= \left\{ \left((y_1 x_2, x_2), [y_1 : 1] \right) \ \middle| \ \begin{array}{l} x_2 - (y_1 x_2)^2 = 0 \\ x_2 \neq 0 \end{array} \right\}$$

$$= \left\{ \left((y_1 x_2, x_2), [y_1 : 1] \right) \ \middle| \ \begin{array}{l} 1 - y_1^2 x_2 = 0 \\ x_2 \neq 0 \end{array} \right\}$$

$$= V(1 - y_1^2 x_2) \cap D(x_2) \subseteq U_2 \cong \mathbb{A}^2.$$

$$\overline{\Gamma_f \cap U_2}^{U_2} = \tilde{X} \cap U_2.$$

$$\Gamma_f \cap U_2 \overset{\text{open}}{\subseteq} \underset{\text{irr}}{V(1 - y_1^2 x_2)} \Rightarrow \overline{\Gamma_f \cap U_2}^{V(1 - y_1^2 x_2)} = V(1 - y_1^2 x_2)$$

$$\xRightarrow{V(1 - y_1^2 x_2) \overset{\text{closed}}{\subseteq} U_2 \cong \mathbb{A}^2} \overline{\Gamma_f \cap U_2}^{U_2} = \overline{\Gamma_f \cap U_2}^{V(1 - y_1^2 x_2)} = V(1 - y_1^2 x_2)$$

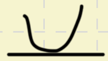
So we have $\begin{cases} \tilde{X} \cap U_1 = V(y_2 - x_1) \\ \tilde{X} \cap U_2 = V(1 - x_2 y_1^2) \end{cases}$

$$\pi^{-1}(a) \cap U_1 = \left\{ (a, [1 : y_2]) \in \tilde{X} \cap U_1 \right\}$$

$$= \left\{ (a, [1 : y_2]) \in V(y_2 - \underset{a_1 = 0}{\underline{\underline{x_1}}}) \right\}$$

$$= \left\{ (0, [1 : 0]) \right\} \underset{\text{as}}{\overset{\text{viewed}}{=}} [1 : 0] \in \mathbb{P}^1$$

So $\pi^{-1}(a)=\{[1:0]\}$ $\quad C\pi^{-1}(a)=V(x_2)\subseteq \mathbb{A}^2$.



切線.

Exp 3: (a 不光滑) $X=V(x_2^2-x_1^2-x_1^3)$.

$$\Gamma_f\cap U_1=\{((x_1,x_1y_2),[1,y_2])\in V(x_2^2-x_1^2-x_1^3)\setminus 0\times \mathbb{P}^1\}$$

$$=\left\{((x_1,x_1y_2),[1,y_2])\ \middle|\ \begin{matrix}(x_1y_2)^2-x_1^2-x_1^3=0\\ x_1\neq 0\end{matrix}\right\}$$

$$\cong V(y_2^2-x_1-1)\cap D(x_1)$$

同理 $\tilde{X}\cap U_1=V(y_2^2-x_1-1)$

同理 $\tilde{X}\cap U_2=V(1-y_1^2-x_2y_1^3)$

$$\pi^{-1}(a)\cap U_1=\{(a,[1:y_2])\in\tilde{X}\cap U_1\}$$

$$=\{(a,[1:y_2])\in V(y_2^2-\underset{=0}{x_1}-1)\}$$

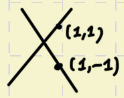
$$=\{(0,[1:1]),(0,[1:-1])\}$$

$$\pi^{-1}(a)\cap U_2=\{(a,[y_1:1])\in V(1-y_1^2-\underset{=0}{x_2}y_1^3)\}$$

$$=\{(0,[1:1]),(0,[1:-1])\}$$

So $\pi^{-1}(a)=\{[1:1],[1:-1]\}\in\mathbb{P}^1$.

$C\pi^{-1}(a)=V(x_2^2-x_1^2)$



- Two props about dimension.

(1) $\pi:\tilde{X}\to X$ be the blow up of an irr aff. var X at $f_1,\dots,f_r\in A(X)$. Then every irr component of exceptional set $\pi^{-1}(V(f_1,\dots,f_r))$ has codim 1 in $\tilde{X}$. (It's called exceptional hypersurface of the blow up)

Idea: 证 $\pi^{-1}(V(f_1,\dots,f_r))$ 的每个 aff. patch 有 codim $=1$ ("pure codim" $=1$)

$$\tilde{X}\subseteq Z=\{(x,[y_1,\dots,y_r])\in X\times\mathbb{P}^{r-1}\mid y_if_j(x)=y_jf_i(x)\ ,\forall i,j\}$$

$$=\bigcup_i U_i\quad\text{where } U_i:=\{y_i=1\}\cap Z=\{(x,[y_1,\dots,1,\dots,y_r])\in X\times\mathbb{P}^{r-1}\mid y_kf_j(x)=y_jf_k(x)\}$$

$$U_i\cap\pi^{-1}(V(f_1,\dots,f_r))=\{(x,[y_1,\dots,y_r])\in\tilde{X}\mid x\in V(f_1,\dots,f_r),\ \boxed{y_if_j(x)=y_jf_i(x),\forall i,j}\}$$

可以扔掉
因为 $x\in V(f_1,\dots,f_r)$

Claim: $U_i\cap\pi^{-1}(V(f_1,\dots,f_r))=V_{U_i}(f_i)$

If the claim is true, we have:

$$\left\{\begin{matrix}X\text{ irr}\Rightarrow\tilde{X}\text{ irr}\Rightarrow\tilde{X}\cap U_i\ (\overset{\text{open}}{\subseteq}\tilde{X})\text{ is irr}\\ f_i\neq 0\end{matrix}\right.\xrightarrow[\text{Thm}]{\text{krull dim}}\forall\text{ irr component of } V_{\tilde{X}}(f_i)\text{ has codim 1 in }\tilde{X}\cap U_i.$$

i.e., $\forall$ irr component of $U_i\cap\pi^{-1}(V(f_1,\dots,f_r))$ has dimension $\dim\tilde{X}\cap U_i-1=\dim\tilde{X}-1$

So $U_i\cap\pi^{-1}(V(f_1,\dots,f_r))$ has pure dimension $\dim\tilde{X}-1$

$\{U_i\cap\pi^{-1}(V(\cdots))\}$ is open cover of $\pi^{-1}(V(\cdots))\Rightarrow\dim\pi^{-1}(V(\cdots))=\sup\{U_i\cap\pi^{-1}(V(\cdots))\}=\dim\tilde{X}-1$.

pf for the claim: $U_i \cap \pi^{-1}(V(f_1,\cdots,f_r)) \subseteq U_i$ 故是U_i上的aff. var.

$\forall (x,y) \in U_i \cap \pi^{-1}(V(f_1,\cdots,f_r))$, we have $x \in V(f_1,\cdots,f_r) \subseteq V(f_i)$.

Hence $U_i \cap \pi^{-1}(V(f_1,\cdots,f_r)) \subseteq V_{U_i}(f_i)$.

$\forall (x,y) \in V_{U_i}(f_i)$, since $f_j(x) = y_i f_j(x) = y_j f_i(x) = 0$.

So $(x,y) \in V(f_1,f_2,\cdots,f_r) \cap U_i$.

(2) $a \in \underset{\text{var}}{\underset{\sim}{X}}$. $\dim C_a X = \operatorname{codim}_X\{a\}$

pf: $X = X_1 \cup X_2 \cup \cdots \cup X_s$ are irr. component decomposition.

Trick: Assume $a \in X_i$, $\forall i$. If $a \notin X_{i_0}$, 由 blow up 的 local 性质忽 可以把 X_{i_0} 舍掉 考虑 $X \setminus X_{i_0}$

$\tilde{X} = \tilde{X}_1 \cup \tilde{X}_2 \cup \cdots \cup \tilde{X}_s \Rightarrow C_a X = C_a X_1 \cup C_a X_2 \cup \cdots \cup C_a X_s$

$\Rightarrow \dim C_a X = \max \dim C_a X_i$

Every irr component of $\pi|_{X_i}^{-1}(a)$ has dimension $\dim X_i - 1$

take cone, 则 dim 刚好加 1

Every irr component of $C_a X_i$ has dimension $\dim C X_i - 1 = (\dim X_i + 1) - 1 = \dim X_i$

So $\dim C_a X = \max \dim C_a X_i = \max \dim X_i = \operatorname{codim}_X\{a\}$

Local dimension $\operatorname{codim}_X\{a\}$ 是 cover a 的 irr comp. 最大维数.

: $\pi^{-1}(a) = \bigcup_i \pi|_{X_i}^{-1}(a) \Rightarrow C\pi^{-1}(a) = \bigcup_i C\pi|_{X_i}^{-1}(a) \Rightarrow C_a X = \bigcup_i C_a X_i$

$(C(A \cup B) = CA \cup CB)$

$\mathbb{P}^{r-1}$ complete so $p: X_i \times \mathbb{P}^{r-1} \to X_i$ is closed. Since $\tilde{X}_i \subseteq X \times \mathbb{P}^{r-1}$ closed, we have $p(\tilde{X}_i)$ closed. $p|_{\tilde{X}_i} = \pi$, so $\pi(\tilde{X}_i)$ closed in X_i.

$X_i \setminus a \overset{\text{open}}{\subseteq} \underset{\text{irr}}{X_i} \Rightarrow X_i \setminus a$ irr and $\overline{X_i \setminus a} = X_i$.

$\Rightarrow X_i = \overline{X_i \setminus a} \subseteq \pi(\tilde{X}_i) \Rightarrow a \in \pi(\tilde{X}_i) \Rightarrow \pi|_{X_i}^{-1}(a) \neq \phi$

$X_i \setminus a \subseteq \underset{\text{closed}}{\pi(\tilde{X}_i)}$

$\Rightarrow C_a X_i = C(\pi|_{X_i}^{-1}(a))$ is well-def.

- Blowing up 可以用来 extend mor.

* X: afl. var. Consider $f: U \longrightarrow \mathbb{P}^{r-1}$, $x \longmapsto (f_1(x) : \cdots : f_r(x))$ where $U = X \setminus V(f_1,\cdots,f_r)$.

We can extend it after blowing up. $\tilde{X}$ is the blow up of X at $\langle f_1,\cdots,f_r \rangle$.

$\tilde{f}: \tilde{X} \longrightarrow \mathbb{P}^{r-1}$ is projection ($\tilde{X} \subseteq X \times \mathbb{P}^{r-1}$) with $\tilde{f}|_U = f$. $\tilde{f}$ is the extension.

$\triangle$ $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in one point is iso to $\mathbb{P}^2$ blow up in two points.

- Tangent space and tangent cone

i) We can approximate a variety by tangent cone, e.g. $\smile$

ii) But in practice one often wants to approximate by a linear space rather than cone.

iii) Tangent space : a linear space, take linear terms of poly.

Tangent cone : 可以不 linear, take initial terms of poly.

• Def: Let $a\in \underset{\text{var}}{X}$. By choosing aff. n.b.h. of a, assume $X\subseteq \mathbb{A}^n$ and $a=0$.

$T_aX := V(f_1 : f\in I(X))$ is called tangent space of X at a.

f_1 : linear term of f.

Note that: ① Why we want $a=0$? $0=a\in X$, 则对 $\forall f\in I(X)$, 有 $f(a)=0$ 即 $f(0)=0$. 于是 $I(X)$ 中任何 poly 常数项为 0.

② $T_aX = V(f_1 : f\in I(X)) = V(f_1 : f\in S)$ where $I(X)=\langle S\rangle$?

$\forall f\in I(X)=\langle S\rangle$, we have $f=\sum_{g_i\in S} a_i g_i \Rightarrow f_1 = \sum a_i(0)\cdot (g_i)_1 + (a_i)_1\cdot \underset{0}{g_i(0)} = \sum a_i(0)\cdot (g_i)_1$

$\Rightarrow f_1 \in \langle (g_i)_1 : g_i\in S\rangle$

③: 注意是 $V(f_1 : f\in I(X))$, 而不是 $V(f_1 : f\in J)$ where $V(J)=X$.

例如 In $\mathbb{A}^1$, $V(x)=V(x^2)=0$, 但 $V(f_1 : f\in\langle x\rangle) = V(f_1 : f\in\{x\}) = V(x) = 0$

$V(f_1 : f\in\langle x^2\rangle) = V(f_1 : f\in\{x^2\}) = V(0) = \mathbb{A}^1$.

在generator上求 已阿.

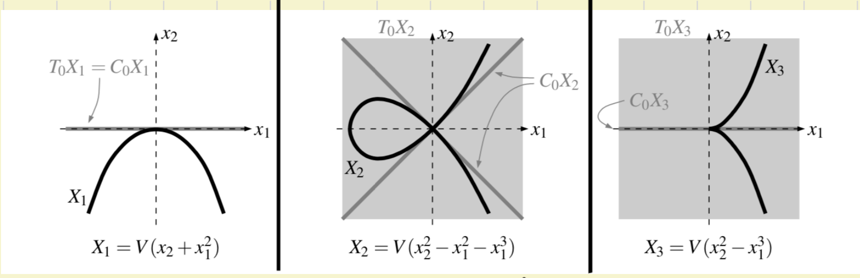
④常数项为0, 则取 initial term 比取 linear term 会出的方程更多, 因此 $T_aX \supseteq C_aX$.

*Tangent space is a linear space containing tangent cone.

于是 $\dim T_aX \geq \dim C_aX = \operatorname{codim}_X\{a\}$.

• Excellent examples:

$X_1 = V(x_2+x_1^2)$	$X_2 = V(x_2^2-x_1^2-x_1^3)$	$X_3 = V(x_2^2-x_1^3)$
$C_0X_1 = V(x_2)$	$C_0X_2 = V(x_2^2-x_1^2)$	$C_0X_3 = V(x_2^2)$
$T_0X_1 = V(x_2)$	$T_0X_2 = V(0) = \mathbb{A}^2$	$T_0X_3 = V(0) = \mathbb{A}^2$



T_aX 可以恰好等于 C_aX.

Tangent space 是包含 tangent cone $V(x_2^2-x_1^2)$ 的 linear space, 所以只能是 $\mathbb{A}^2$.

并不是所有 T_aX 都是 C_aX 张成的.

• Tangent space 的内蕴描述.

Δ Lemma: $X\subseteq \mathbb{A}^n$ aff. var. $a=0\in X$. $I(a) = \langle \underset{\text{等价类}}{\bar{x}_1}, \dots, \bar{x}_n\rangle \trianglelefteq A(X) = k[x_1,\dots,x_n]/I(X)$

Then we have $I(a)/I(a)^2 \cong \mathrm{Hom}_k(T_aX, k)$

* $\mathrm{Hom}_k(T_aX, k)$ is the dual of tangent space.

Slogan: vector space T_aX dual to $I(a)/I(a)^2$.

pf: Idea: Check "$\ker\varphi = I(a)^2$". Routine check

Let $\varphi: I(a) \to \mathrm{Hom}_k(T_aX, k)$, w.t.s $\begin{cases} (1)\ \varphi \text{ well-defined} \\ (2)\ \varphi \text{ surj.} \\ (3)\ \ker\varphi = I(a)^2 \end{cases}$

$\bar{f} \longmapsto f_1|_{T_aX}$

Trick: ① 描述 $I(a)$ 中的元素只需要放到 $A(X)=k[x_1,\dots,x_n]/I(X)$ 中

$\bar{f} \in I(a) \trianglelefteq A(X) = k[x_1,\dots,x_n]/I(X)$. $f \in k[x_1,\dots,x_n]$, $\bar{f}$ 是等价类.

② $\varphi(\bar{f}) = ?$ $T_aX = V(f_1 : f \in I(X)) \subseteq \mathbb{A}^n$. $\mathrm{Hom}_k(T_aX, k)$ 是 T_aX 上的 linear function, 即 linear poly. 这里唯一出现 poly 是 f, 因此构造 linear poly f_1.

(1) Show φ well-defined.

For $\bar{f} = \bar{g} \in I(a)$, it means $f - g \in I(X)$. $\xRightarrow{T_aX = V(h_1 : h \in I(X))}$ $f_1 - g_1|_{T_aX} = 0$. $\Rightarrow f_1|_{T_aX} = g_1|_{T_aX}$

(2) Show φ is surj. Let $h \in \mathrm{Hom}_k(T_aX, k)$, i.e., f is a linear function on T_aX.

$T_aX \subseteq \mathbb{A}^n$. We extend f to a linear function on $\mathbb{A}^n$. We have $\bar{f} \in I(a) \trianglelefteq A(X)$ be preimg.

(3) Show $\ker\varphi = I(a)^2$.

"$\subseteq$" $\underset{\forall h \in \ker\varphi}{\underline{\forall \bar{h} \in I(a) \text{ with } h_1|_{T_aX} = 0}}$, since $h_1|_{V(f_1 : f \in I(X))} = 0$, 下面的 claim is reasonable.

Claim: $\exists g \in I(X)$ s.t. $g_1 = h_1$. If the claim is true, then $g-h$ has no const or linear term. Hence we have $\bar{h} = \underline{\overline{h-g}} \in I(a)^2$

$h-g$ 只有 ≥ 2 次 terms.

pf the claim: Dimension is really important when consider vector space.

Let $S = \{f_1 \mid f \in I(X)\}$ and $W = \{\text{linear forms vanishing at } T_aX\}$. It suffices to show $S = W$. (则 $h_1 \in W$ 必可写成 $g_1, g \in I(X)$.) $T_aX = V(S)$, so $W \supseteq S$. We view S as sublinear space of $k[x_1,\dots,x_n]$ with dimension k, i.e., there're k linear equs. So $T_aX = V(S)$ has dimension $n-k$. So $\underline{\dim W = n-(n-k) = k}$.

$\dim W = \dim S$ and $S \subseteq W$. So $S = W$.

A is a linear transformation.

$T_aX = \ker A$. $\mathrm{rank} A = n - \dim\ker A = n-(n-k) = k$.

"$\supseteq$" Let $\bar{f}, \bar{g} \in I(a)$, then $f(0) = g(0) = 0$ (f 与 g vanish at $a=0$)

For $\overline{fg} \in I(a)^2$, $\varphi(\overline{fg}) = (fg)_1 = \underset{0}{\underline{f(0)}} \cdot g_1 + f_1 \cdot \underset{0}{\underline{g(0)}} = 0$. So $I(a)^2 \subseteq \ker\varphi$.

△ Intrinsic description of tangent space

To obtain intrinsic description, we need transfer $A(X)$ to $\mathcal{O}_{X,a}$

For general variety would require choice of an aff. coord that sends a to origin. (any point can have coord. $(x_1,\dots,x_n)$ when choosing an aff. patch.)

Def 3.19 in Gathmann: $\mathcal{O}_{X,a} \cong A(X)_{I(a)}$.

Besides $\mathcal{O}_{X,a}$ is a local ring with unique max ideal

$I_a := \{\overline{(U,\varphi)} \in \mathcal{O}_{X,a} \mid \varphi(a) = 0\}$

$\cong I(a)_{I(a)} = \{\frac{g}{f} \mid g \in I(a), f \notin I(a)\}$

$\mathcal{O}_{X,a}$ is independent of choice of aff. var.

prop: $I(a)/I(a)^2 \cong I_a/I_a^2$.

pf. $\forall f \notin I(a)$, we have $f(a) \neq 0$. Trick: $f(a) \neq 0$ means $\frac{1}{f(a)}$ make sense.

$\varphi: A(X)/I(a) \longrightarrow k$ is an iso. (Intuitively, $A(X)/I(a) = A_X(a)$ is functions on a, which is k. rigorous check is also easy)

$\bar{h} \longmapsto h(a)$

$f(a) \neq 0 \Rightarrow 0 \neq \frac{1}{f(a)} \in k \overset{\text{By } \varphi}{\Rightarrow}$ An element denoted by $\frac{1}{f}$ in $A(X)/I(a)$ corresponds to $\frac{1}{f(a)} \in k$

Claim: $I(a)/I(a)^2 \cong S^{-1}(I(a)/I(a)^2)$, where $S = A(X) \setminus I(a)$.

pf the claim: For any $\frac{g}{f} \in S^{-1}(I(a)/I(a)^2)$, i.e., $\begin{cases} f \in A(X) \setminus I(a) \overset{f \notin I(a)}{\Longrightarrow} \exists \frac{1}{f} \in A(X)/I(a) \text{ corresponds to } \frac{1}{f(a)} \in k. \\ g \in I(a)/I(a)^2 \end{cases}$

$\frac{g}{f} \overset{?}{=} \frac{1}{f(a)} g \in I(a)/I(a^2)$ so localization at S doesn't change $I(a)/I(a)^2$, since elements of S already invertible.

? S 可逆

$*$ $T_aX = \left(I(a)/I(a)^2\right)^* \cong \left(I_a/I_a^2\right)^*$ where $I_a = I(a)_{I(a)} = S^{-1}I(a)$ is maximal ideal of $\mathcal{O}_{X,a}$. Hence T_aX is independent of choice of aff. patch.

- Smooth

Δ Motivation —— When C_aX and T_aX agree.

$\begin{cases} \text{tangent cone } C_aX: \text{ may Not linear, dimension} = \operatorname{codim}_X\{a\}. \\ \text{tangent space } T_aX: \text{ Linear space, dimension may bigger } \operatorname{codim}_X\{a\} \end{cases}$

$\Rightarrow$ Hence, we should pay attention to the case when two notations agree, i.e., X can be approximated around a by a linear space whose dimension is $\operatorname{codim}_X\{a\}$.

Δ Def: point $a \in X$ is called smooth, regular or nonsingular if $T_aX = C_aX$. Otherwise is called a singular point of X.

Δ $a \in X$ is a point. TFAE

(a) a is smooth

(b) $\dim T_aX = \operatorname{codim}_X a$

(c) $\dim T_aX \leq \operatorname{codim}_X a$

pf: (a)$\Rightarrow$(b) a sm $\Rightarrow T_aX = C_aX \Rightarrow \dim T_aX = \dim C_aX = \operatorname{codim}_X\{a\}$

(b)$\Rightarrow$(c) obviously.

(c)$\Rightarrow$(a) $\begin{cases} T_aX \supseteq C_aX \Rightarrow \dim T_aX \geq \dim C_aX = \operatorname{codim}_X a \\ T_aX \leq \operatorname{codim}_X a \end{cases} \Rightarrow \dim T_aX = \operatorname{codim}_X a = \dim C_aX$

Since T_aX is irr, $T_aX = C_aX$.

- Useful prop: a smooth, 则至多只有一片 irr. 分支 cover a.

Δ $T_aX \cong (I_a/I_a^2)^* \Rightarrow \dim I_a/I_a^2 = \dim (I_a/I_a^2)^* = \dim T_aX \overset{a \text{ sm}}{=\!=} \operatorname{codim}_X a$

where I_a is the max ideal of local ring $\mathcal{O}_{X,a}$. Ring $\mathcal{O}_{X,a}$ with this prop is called regular local ring. Hence a sm $\Rightarrow \mathcal{O}_{X,a}$ regular $\underset{\text{Commutative alg.}}{\Longrightarrow} \mathcal{O}_{X,a}$ integral domain

Δ $\mathcal{O}_{X,a}$ is a domain means the variety is $\underset{X \text{ 只有一个 irr component cover } a.}{\underline{\text{locally irr}}}$ at every sm a. (Intuitively)

(两片 irr component 交点, 都是 singular)

- Aff. Jacobi criterion $a \in X$ aff. Let $I(X) = \langle f_1, \cdots, f_r \rangle$. Then X is sm at a iff $\operatorname{rank} J \geqslant n - \operatorname{codim}_X a$ (actually, when a sm $\operatorname{rank} J = n - \operatorname{codim}_X a$.)

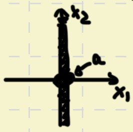
pf: We can always assume $a=0$ when X aff. (Let $y = x - a$, a shift). For any generator f_j, $T_a X = V((f_j)_1 : j = 1, \cdots, r)$. By Taylor expansion, $f_j = \sum_i \frac{\partial f_j}{\partial x_i} \cdot x_i$

Hence $T_a X = \{ x \mid \sum \frac{\partial f_j}{\partial x_i} x_i = 0, \forall j = 1, 2, \cdots, r \} = \ker J$.

a sm $\Rightarrow \dim \ker J = \dim T_a X \underset{(\text{or } =)}{\leqslant} \operatorname{codim}_X a$. Since $\dim \ker J = n - \operatorname{rank} J$, $\operatorname{rank} J \underset{(\text{or } \geqslant)}{=} n - \operatorname{codim}_X a$

Note that it's not true for generators $f_1', \cdots, f_r'$ for $X = V(f_1', \cdots, f_r')$

Suppose $X = V(x_1^2)$ $\qquad \operatorname{rank} J = \operatorname{rank}(2x_1 \ \ 0)|_0 = 0 \neq 2 - 1 = 1$



$\operatorname{codim}_X a = 1$

- $V(f_1, \cdots, f_r)$ version of Jacobi criterion.

$a \in X = V(f_1, \cdots, f_r)$ $\qquad J_{ij} = \frac{\partial f_i}{\partial x_j}$

(a) If $\operatorname{rank} J \geqslant n - \operatorname{codim}_X a$, then a sm (反过来不一定)

(b) If $\operatorname{rank} J = \underset{J\text{满秩}}{r}$ then $\begin{cases} X \text{ is sm at } a \\ \operatorname{codim}_X a = n - r. \end{cases}$

Pf:

(a) extend generator $J = \langle f_1, \cdots, f_r \rangle$ to generators of $\sqrt{J} = I(V(J)) = I(X)$ as $f_1, \cdots, f_r, f_{r+1}, \cdots, f_s$.

Then $\operatorname{rank} \left(\frac{\partial f_i}{\partial x_j} \right)_{\substack{i \in [\![1,s]\!] \\ j \in [\![1,n]\!]}} \geqslant \operatorname{rank} \left(\frac{\partial f_i}{\partial x_j} \right)_{\substack{i \in [\![1,r]\!] \\ j \in [\![1,n]\!]}} = \operatorname{rank} J \underset{\text{By assumption}}{\geqslant} n - \operatorname{codim}_X a \Rightarrow a$ sm

(b) $\operatorname{codim}_X a = \max_{\text{irr } X_i \ni a} \dim X_i$. Trick: $X = V(f_1, \cdots, f_r) \Rightarrow \forall$ irr component has dimension $n - r$.

So $\operatorname{codim}_X a \geqslant n - r \xRightarrow{\operatorname{rank} J = r} \operatorname{codim}_X a \geqslant n - \operatorname{rank} J \underset{\text{移项}}{\Rightarrow} \operatorname{rank} J \geqslant n - \operatorname{codim}_X a \xRightarrow{\text{By (a)}} a$ sm

$$\underbrace{\operatorname{rank} \left(\frac{\partial f_i}{\partial x_j} \right)_{s \times n}}_{\substack{\| \, a \text{ sm} \\ n - \operatorname{codim}_X a}} \geqslant \operatorname{rank} \left(\frac{\partial f_i}{\partial x_j} \right)_{r \times n} \geqslant n - \operatorname{codim}_X a \Rightarrow \overset{r}{\overset{\|}{\operatorname{rank}}} \underbrace{\left(\frac{\partial f_i}{\partial x_j} \right)_{r \times n}}_{\substack{\| \\ J}} = n - \operatorname{codim}_X a$$

$\Rightarrow \operatorname{codim}_X a = n - r$

* Relation between 隐函数定理.

$a \in X = V(f_1, \cdots, f_r)$ with $\operatorname{rank} \left(\frac{\partial f_i}{\partial x_j} \right)_{ij} = r$. So we can view X as a graph of continously differentiable function (it doesn't have "corners"). The following example show difference between Implicit Function thm.

Exp: $f(x_1, x_2) = x_2 - x_1^2$. $V(f)$ at $(1, 1)$ is sm $(\operatorname{rank}(-2 \ \ 1) = 1 = 2 - \overset{1}{\overset{\|}{\operatorname{codim}_X a}} = 1)$

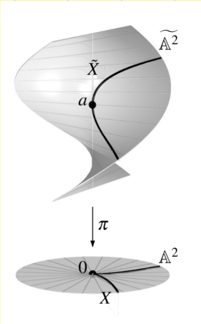
But $f(x_1, x_2) = 0$ cannot be solved for x_1 by a regular function — It can only solved by continous function $x_1 = \sqrt{x_2}$

- Blowing up makes singular points "nicer"

Consider $X = V(x_2^2 - x_1^3)$. ① We've prove $0 \in X$ singular. ($T_aX \neq C_aX$)

② $\operatorname{rank}(-3x_1^2 \quad 2x_2) \geq 2-1=1 \Leftrightarrow (x_1,x_2) \in \mathbb{A}^2 \setminus 0$

So $\mathbb{A}^2$ is sm at $x \neq 0$.



Consider blow up $\tilde{X}$ on aff. patch $\{((x_1, x_1y_2), (1:y_2))\}$, by Exe 9.22

$\tilde{X}$ is given by $V(g(x_1,x_2))$ where $g = y_2^2 - x_1$. $\left(\frac{\partial g}{\partial x_1} \ \frac{\partial g}{\partial y_2}\right) = (-1 \ 2y_2)|_0 = (-1 \ 0)$

so $\tilde{X}$ is sm at a.

Slogan: Good blow up remove all singularities.

- Projective version Jacobi criterion. (Same as aff. version)

$I(X) = \langle f_1, \cdots, f_r \rangle$. Then X is sm at $a \Leftrightarrow r \times (n+1)$ matrix $\left(\frac{\partial f_i}{\partial x_j}(a)\right)_{i,j}$ has rank $\geq n - \operatorname{codim}_X a$

△Exp: Fermat hypersurface $X = V_p(x_0^d + \cdots + x_n^d) \subseteq \mathbb{P}^n$.

X is sm for all n, d, k.

$J = (d x_0^{d-1} \ \cdots \ d x_n^{d-1})$.

i) If $\operatorname{char} k \nmid d$, $\operatorname{rank} J = 1$. X is sm.

ii) If $\operatorname{char} k \mid d$. Let $\operatorname{char} k = p$ and say $\begin{cases} d = p^r k \\ d \nmid k \end{cases}$.

Since $x_0^d + \cdots + x_n^d = (x_0^k + \cdots + x_n^k)^{p^r}$, we have $I(X) = \langle x_0^k + \cdots + x_n^k \rangle$.

$J = (k x_0^{k-1}, \cdots, k x_n^{k-1})$. So $\operatorname{rank} J = 1$. X is sm.

- The set of sm pts in a variety X is open

Step 1: $a \in X$ sm, we w.t.s. there is a n.b.h. of a which all pts in it are sm.

*Idea: 令 N 是 a 的邻域且由 sm pts 组成. 证明 $x \in N \Leftrightarrow x$ 满足开条件

a sm $\Rightarrow$ 只有唯一 irr aff. component cover a, 记作 U

say $I(U) = \langle f_1, \cdots, f_r \rangle$. So by Jacobi Criterion, let $J = \frac{\partial f_i}{\partial x_j}$

$\operatorname{rank} J \geq n - \operatorname{codim}_U a \Leftrightarrow \exists (n - \operatorname{codim}_U a)$-minors with determinant unzero.

$\det \neq 0$ 是开条件 (write as $D(\cdots)$, where $D(\cdots)$ is an open set)

Step 2: 还要证明一定存在 sm pts. Sketch: 转化到 hypersurface 情形.