Stable homotopy theory

§ Stable is a phenomenon in homotopy theory.

[Exp] S^{n-1} has $\rho(n)-1$ linearly independent vector fields.
$\rho(n)$ is computed by expressing n as $n=(2a+1)2^b$ and $b=c+4d$ with $0\le c<4$
then setting $\rho(n)=2^c+8d$.

[Exp] There is iso $\Omega_n \cong \pi_{m+n}(\underbrace{MSO(m)}_{\text{Thom complex}})$ for $m>n+1$

In above two examples, something holds/occurs in all sufficiently large dimension.
The meaning of "sufficiently large" usually depends on the connectivity of spaces.
The following is an example.

[Construction] $\Sigma : [X,Y] \longrightarrow [\Sigma X, \Sigma Y]$

$$[f: X\to Y] \longmapsto [1\wedge f : S^1\wedge X \to S^1\wedge Y]$$

[Def] When suspension $\Sigma : [X,Y]\to[\Sigma X,\Sigma Y]$ is an iso, maps in $[X,Y]$ are called stable.

[Thm] If Y is $n-1$-connected, then Σ is $\begin{cases} \text{surj} & \dim X = 2n-1 \\ \text{bij} & \dim X < 2n-1 \end{cases}$

[Exp] (Use this example to remember $n-1$, $2n-1$ in above thm)

$$[\underset{3=2n-1}{S^3}, \underset{n=2}{S^2}] = \mathbb{Z}\eta, \quad [\underset{4<2n-1}{S^4}, \underset{n=3}{S^3}] = \mathbb{Z}/2, \quad [S^5, S^4] = \mathbb{Z}/2.$$

so $\Sigma: [S^3,S^2]\to[S^4,S^3]$ is surj and

$\Sigma: [S^4,S^3]\longrightarrow[S^5,S^4]$ is bij

[Rmk] We've said the meaning of "sufficiently large" depends on connectivity.
It means:

$$[X,Y]\to[\Sigma X,\Sigma Y]\to[\Sigma^2X,\Sigma^2Y]\to\cdots\to[\Sigma^kX,\Sigma^kY]\xrightarrow{\cong}[\Sigma^{k+1}X,\Sigma^{k+1}Y]$$
$$\xrightarrow{\cong}[\Sigma^{k+2}X,\Sigma^{k+2}Y]\xrightarrow{\cong}\cdots \text{ (All iso's)}$$

This occurs because:

Let X be m-conneeted and Y be n-connected.
Σ^kX is $m+k$-connected and Σ^kY is $(n+k)$-connected ($\pi_{m+k}(\Sigma^kX)=\pi_m(X)$)
$m+k<2(n+k)-1$ holds for all sufficiently large k ($m+k\sim k$, $2(n+k)-1\sim 2k$)
Hence $[\Sigma^kX,\Sigma^kY]\xrightarrow{\Sigma}[\Sigma^{k+1}X,\Sigma^{k+1}Y]$ is iso for all sufficiently large k
and clearly it depends on connectivity m and n.

§ Spectra

• Big picture $\begin{cases} \text{A useful tool for study stable pheonomena in homotopy theory} \\ \text{In cat of spectra, suspension is an equivalence} \end{cases}$

[Def] A spectrum E is a seq. of spaces $\{E_n\}_{n\in\mathbb{Z}}$ with a basepoint together with maps $\varepsilon_n : \Sigma E_n \to E_{n+1}$.

[Rmk] Index n may vary over $\mathbb{Z}_{\geq 0}$

[Construction] By adjunction, $\varepsilon_n : \Sigma E_n \to E_{n+1}$ adjoint to $\varepsilon_n' : E_n \to \Omega E_{n+1}$. If E_n is connected, $\varepsilon_n'(E_n)$ is connected in ΩE_{n+1}, i.e., $\varepsilon_n'(E_n)$ lies in the connected component containing base point.

[Def] Denote $\Omega_0 E_{n+1}$ be the connected component of ΩE_{n+1} containing base point. Then $\varepsilon_n'(E_n)$ lies in $\Omega_0 E_{n+1}$ if E_n is connected.

[Def] Let E, F be two spectra. A map of degree r between spectra E and F are a seq of maps $f_n : E_n \to F_{n-r}$ s.t.

$$\begin{tikzcd} \Sigma E_n \arrow[r, "\Sigma f_n"] \arrow[d, "\varepsilon_n"'] & \Sigma F_{n-r} \arrow[d, "\varepsilon_{n-r}"] \\ E_{n+1} \arrow[r, "f_{n+1}"'] & F_{n-r+1} \end{tikzcd}$$

diagram commutes strictly.
(not up to homotopy!)

A map of degree 0 $f : E \to F$ is also called a map between spectra.

• Omega spectra

[Def] Spectra where ε_n' are weak equiv are called omega spectra.

[Exp] Eilenberg-Maclane spectrum for grp π, denoted by $H\pi$, is an omega spectra.

• Thom spectra: $MO, MU, MSO, \dots$

[Def] Let $E \to X$ be a vector bdl with continously varying norm on the fibers. Let $D(E) := \{e \in E \mid |e| \leq 1\}$, $S(E) = \{e \in E \mid |e| = 1\}$. Let $Th(E) := D(E)/S(E)$ be the Thom space of E.

[Fact] Let X be a compact CW-complex, $E \to X$ be vector bdl. $Th(E)$ is homotopy equiv to E^+, the one pt compactification of E.

[Prop] Let $\underline{1}$ be the trivial real vector bdl of rank 1 on X. Then

$$Th(\underline{1} \oplus E) \cong S^1 \wedge Th(E) = \Sigma Th(E)$$

pf: [Fact] By choosing a sup metric, $D(\underline{1} \oplus E) = [0,1] \times D(E)$

$S(\underline{1} \oplus E) = \partial D(\underline{1} \oplus E) = \{0,1\} \times D(E) \cup [0,1] \times \partial D(E) = \{0,1\} \times D(E) \cup [0,1] \times S(E)$

Then $Th(\underline{1} \oplus E) = \dfrac{D(\underline{1} \oplus E)}{S(\underline{1} \oplus E)} = \dfrac{\overbrace{[0,1] \times D(E)}^{=:U}}{\underbrace{\{0,1\} \times D(E) \cup [0,1] \times S(E)}_{=:D}}$

$= \dfrac{S^1 \times D(E)}{* \times D(E) \cup S^1 \times S(E)}$ (all first component in the product spaces quotient $\{0,1\}$)

$$= \frac{S^1 \times Th(E)}{* \times Th(E) \cup S^1 \times *}$$ (all second component in the product spaces quotient $S(E)$)

$$= \frac{S^1 \times Th(E)}{Th(E) \vee S^1} = S^1 \wedge Th(E)$$

[Construction] (Spectrum MO)

Let $\begin{array}{c} EO(n) \\ \downarrow{\scriptstyle \xi_n} \\ BO(n) \end{array}$ be the universal bdl (also called the tautological bdl over $BO(n)$).

Then any real n-vector bdl over Y is a pullback of ξ_n under $Y \to BO(n)$.

$\underline{1} \oplus \xi_n$ is a real $(n+1)$-vec bdl over $BO(n)$ with $f: \underline{1} \oplus \xi_n \to \xi_{n+1}$

$$\begin{tikzcd} \underline{1} \oplus EO(n) \arrow[r] \arrow[d, "\underline{1}\oplus\xi_n"'] & EO(n+1) \arrow[d, "\xi_{n+1}"] \\ BO(n) \arrow[r] & BO(n+1) \end{tikzcd}$$

Applying Th, we have a map

$$\varepsilon_n = f_*: \underbrace{Th(\underline{1} \oplus \xi_n)}_{\Sigma Th(\xi_n)} \to Th(\xi_{n+1})$$

Define a spectrum MO with nth space $Th(\xi_n)$ and structure map $\varepsilon_n: Th(\underline{1} \oplus \xi_n) \to Th(\xi_{n+1})$

[Construction] (Spectrum MU)

Let $\underline{1}$ be trivial complex vector bdl of rank 1.

Let $\gamma_n: EU(n) \to BU(n)$ be universal bdl over $BU(n)$. Analogously,

$(MU)_{2n} := Th(\gamma_n)$ since γ_n is a real $2n$-vect bdl.

$(MU)_{2n+1} := \Sigma Th(\gamma_n)$. The structure maps are

$\varepsilon: Th(\gamma_n) \to \Sigma Th(\gamma_n)$ and $\Sigma Th(\gamma_n) = Th(\underline{1} \oplus \gamma_n) \to Th(\gamma_{n+1})$

(suspension) ($\underline{1} \oplus \gamma_n$ is a complex $(n+1)$-vector bdl, this map induced by pullback.)

[Construction] (Space MU)

If Σ is an equiv, we can construct a spectrum $\overline{MU}$ by $\overline{MU}_n = \Sigma^{-2n} Th(\gamma_n)$ with str map ε_n.

There is a canonical map:

$$\overline{MU}_{n-1} \cong \Sigma^{-2n+2} Th(\gamma_{n-1}) \cong \Sigma^{-2n} Th(\gamma_{n-1} \oplus \underline{1}) \xrightarrow{\varepsilon_n} \Sigma^{-2n} Th(\gamma_n) \cong \overline{MU}_n$$

Hence we can define space $MU = \operatorname{colim}_n \overline{MU}_n$.

§ Stable homotopy cat

- The mor in stable homotopy cat are something like homotopy classes of maps of spectra.
- There are many constructions of stable homotopy cat, here we provide one is constructed by: ① Form a cat of spectra ② Change maps to something like homotopy classes of maps.
- drawbacks of this construction of stable homotopy cat: Stable homotopy cat needs associative and commutative smash product, but in step ① of our construction, we start from a cat of spectra which doesn't admit an ass. and comm. $\wedge$.

[Def] E is a CW-spectrum if $\begin{cases} E_n \text{ is a CW complex with base point} \\ \varepsilon_n: \Sigma E_n \to E_{n+1} \text{ is an iso from } \Sigma E_n \text{ to a subcomplex of } E_{n+1} \end{cases}$

[Fact] We have spectrum version of Whitehead thm: any spectrum is weakly equiv to a CW-spectrum.

Hence, we restrict attention to CW-spectra.

[Def] (subspectrum) A <u>subspectrum</u> A of a CW-spectrum E is a spectrum with $A_n \subset E_n$ as a subcomplex. A is said to be <u>cofinal</u> in E if for each n and finite subcomplex $K \subset E_n$, $\exists\, m$ s.t. $\Sigma^m K$ maps into A_{m+n} under canonical map $\Sigma^m E_n \xrightarrow{\Sigma^{m-1}\varepsilon_n} \Sigma^{m-1}E_{n+1} \xrightarrow{\Sigma^{m-2}\varepsilon_{n+1}} \Sigma^{m-2}E_{n+2} \to \cdots \to E_{m+n}$

[Rmk] For any n, for any $K \subset E_n$, maybe $K \not\subseteq A_n \subseteq E_n$. But we study something occur for all sufficiently large degree, which means we only require $\exists m$ s.t. $\Sigma^m E_n \to E_{m+n}$ takes $\Sigma^m K$ to A_{m+n}.

[Rmk] $\exists m$ s.t. $\begin{array}{ccc} \Sigma^m E_n & \to & E_{m+n} \\ \cup\!\!| & & \cup\!\!| \\ \Sigma^m K & \to & A_{m+n} \end{array}$, then for any $k \geq m$, we have $\begin{array}{ccc} \Sigma^k E_n & \to & E_{k+n} \\ \cup\!\!| & & \cup\!\!| \\ \Sigma^k K & \to & A_{k+n} \end{array}$.

For example, $\Sigma^{m+1}E_n \to E_{m+n+1}$ decompose as

$$\begin{array}{ccccc} \Sigma(\Sigma^m E_n) & \to & \Sigma E_{m+n} & \xrightarrow{\varepsilon_{m+n}} & E_{m+n+1} \\ \cup\!\!| & & \cup\!\!| & & \cup\!\!| \\ \Sigma^{m+1}K = \Sigma(\Sigma^m K) & \to & \Sigma A_{m+n} & \to & A_{m+n+1} \end{array}$$

[Prop] Intersection of cofinal subspectrum is a cofinal subspectrum.

Pf: Let $A, B \subseteq E$ be two cofinal subspectra. $A_n \cap B_n \subseteq E_n$.

For any n and $K \subseteq E_n$, $\exists\, m_A, m_B$ s.t. $\Sigma^{m_A} K \to A_{m_A+n}$ and $\Sigma^{m_B} K \to B_{m_B+n}$.

This notation means $\Sigma^{m_A} K$ taken to A_{m_A+n} by $\Sigma^{m_A} E_n \to E_{m_A+n}$

Pick $m = \max\{m_A, m_B\}$, we have $\Sigma^m K \to A_{m+n}$, $\Sigma^m K \to B_{m+n}$, so $\Sigma^m K \to (A \cap B)_{m+n}$.

[Def] Let E' and E'' be two cofinal subspectra of E and $\begin{array}{l} E' \xrightarrow{f'} F \\ E'' \xrightarrow{f''} F \end{array}$ be two maps. Say f' and f'' are equiv if $\exists$ a cofinal subspectrum E''' contained in E' and E'' s.t. $f'|_{E'''} = f''|_{E'''}$.

[Rmk] The relation f' equiv to f'' is an equiv relation.

① Since $E' \subseteq E'$ is a cofinal subspectrum, $f' \sim f'$. ② $f' \sim f'' \Rightarrow f'' \sim f'$, obviously.

③ $f': E' \to F$, $f'': E'' \to F$, $f''': E''' \to F$ are three maps from cofinal subspectrum E', E'', E'''.

Then $\exists$ cofinal subspectrum $E_{12} \subseteq E'$, $E_{12} \subseteq E''$ s.t. $f'|_{E_{12}} = f''|_{E_{12}}$.

$\exists$ cofinal subspectrum $E_{23} \subseteq E''$, $E \subseteq E'''$ s.t. $f''|_{E_{23}} = f'''|_{E_{23}}$.

So $f'|_{E_{12} \cap E_{23}} = f''|_{E_{12} \cap E_{23}} = f'''|_{E_{12} \cap E_{23}} \Rightarrow f' \sim f'''$

(Intersection of cofinal subspectrum is a cofinal subspectrum)

[Def] An equiv class of a map from a cofinal subspectrum of E to F will be called a <u>pmap</u> from E to F.

[Rmk] Analog to rational map which is only defined on the open set of a space. An equiv class is defined on cofinal subspectrum.

[Exp] Define $\mathbb{S} := \Sigma^\infty S^0$, i.e., $\mathbb{S}_n = \Sigma^n S^0 = S^n$, called sphere spectrum. Define $K \subset \mathbb{S}$ with $K_n = *$ for $n \leq 2$ and $K_n = S^n$ for $n \geq 3$. Let $\eta: S^3 \to S^2$ be the Hopf map. For $n \geq 0$, $\Sigma^n \eta$ defines $K_{n+3} \to \mathbb{S}_{n+2}$. For $n < 0$ $K_{n+3} = * \to \mathbb{S}_{n+2}$ is the unique based map. K_n is a cofinal subspectrum, and these maps are pmap of degree 1 from $\mathbb{S}$ to $\mathbb{S}$.

[prop] Composition of pmaps is well-defined.

pf: Let $f: E \to F$ be a map of spectra and F' be a cofinal subspectrum of F.

<u>Step1.</u> Show there exists a maximal subspectrum $E' \subseteq E$ with E' maps to F'.

For $f_n: E_n \to F_n$, $\exists$ a maximal CW complex E_n' s.t. $f_n(E_n') \subseteq F_n'$. To show

We claim $(E')_n = E_n'$ with str map be restriction from E is a spectrum.

Indeed, since f is a map of spectra, we have

$$\begin{tikzcd}
\Sigma E_n' \arrow[ddd, "E_n' \text{ maps to } F_n'"'] & & \\
 & \Sigma E_n \arrow[r, "\varepsilon_n"] \arrow[d, "\Sigma f_n"'] & E_{n+1} \arrow[d, "f_{n+1}"] \\
 & \Sigma F_n \arrow[r, "\varepsilon_n"'] & F_{n+1} \\
\Sigma F_n' \arrow[rr] & & F_{n+1}'
\end{tikzcd}$$

F_n' is a subspectrum

Hence $\varepsilon_n(\Sigma E_n')$ maps to F_{n+1}' by community. Since E_{n+1}' is the maximal complex that can maps to F_{n+1}', we have $\varepsilon_n(\Sigma E_n')$ maps to E_{n+1}', i.e., restriction on $\Sigma E_n'$ $\varepsilon_n: \Sigma E_n' \to E_{n+1}'$ is well-defined.

Hence E' is the maximal spectrum that maps to the given subspectrum F'.

<u>Step2.</u> Show E' is cofinal

For $K \subseteq E_n$ be a finite CW complex, K is compact. So $f_n(K)$ is compact and thus is contained in a finite CW complex K_F in F_n ①. F is cofinal, so $\exists\, m$ s.t. $\Sigma^m K_F$ maps to F_{m+n}' under map $\Sigma^m F_n \to F_{m+n}$ ②. With

$$\begin{tikzcd}
\Sigma^m K \arrow[ddd, "①"] & & \\
 & \Sigma^m E_n \arrow[r, "\varepsilon'"] \arrow[d] & E_{n+m} \arrow[d] \\
 & \Sigma^m f_n \arrow[r] & F_{n+m} \\
\Sigma^m K_F \arrow[rr, "③"'] & & F_{n+m}'
\end{tikzcd}$$

,

we have $\varepsilon'(\Sigma^m K)$ maps to F_{n+m}, i.e.,

$\varepsilon'(\Sigma^m K) \subseteq E_{n+m}'$. Hence E' is cofinal.

Step 3. Show composition of pmap is well-defined. (This part has same notation but different meaning.)

Let $f: E \to F$, $g: F \to H$ be two pmaps defined on cofinal E', F' respectively. so f, written as $f: E' \to F$, is a map between spectra (Note that map between spectra should define on all E'). Then we can use step1 & step2, $\exists$ a cofinal subspectrum E'' of E', s.t. E'' maps to F' under f. It's easy to show cofinal subspectrum of a cofinal subspectrum is a cofinal subspectrum (Analog to open set of an open set is an open set), so E'' is a cofinal subspectrum of E that maps to F'. Hence gf defined on E'' is well-defined.

[Construction] (Homotopy class of a pmap)

$X_+ := X \amalg *$, a disjoint union of a pt. Define spectrum $Cyl(E)$ by $Cyl(E)_n := [0,1]_+ \wedge E_n$ with str maps $S^1 \wedge [0,1]_+ \wedge E_n \to [0,1]_+ \wedge E_{n+1}$

which is the composition $S^1 \wedge [0,1]_+ \wedge E_n \xrightarrow{\text{flip}} [0,1]_+ \wedge S^1 \wedge E_n \xrightarrow{id \wedge \varepsilon_n} [0,1]_+ \wedge E_{n+1}$

Define $i_0: E \to Cyl(E)$ by $(i_0)_n : E_n \cong \{0\}_+ \wedge E_n \hookrightarrow [0,1]_+ \wedge E_n$

and $i_1: E \to Cyl(E)$ by $(i_1)_n : E_n \cong \{1\}_+ \wedge E_n \hookrightarrow [0,1]_+ \wedge E_n$

Two pmaps $f, g: E \to F$ are homotopic if $\exists$ pmap $H: Cyl(E) \to F$ s.t. $Hi_0 = f$ and $Hi_1 = g$. It can be proved it's a equiv relationship.

[Def] A mor $E \to F$ in the stable homotopy cat from a CW-spectrum E to F is a homotopy class of pmaps. A mor of degree r is a homotopy class of a degree r pmap. Denote $[E,F]$ be mors from E to F in stable homotopy cat and let $[E,F]_r$ denote mors of degree r.

[Rmk] We'll see $[\Sigma^\infty X, H\mathbb{Z}]_{-r} \cong H^r(X, \mathbb{Z})$. mors of degree r appear in cohomology.

§ Generalized cohomology

[Lemma] Let F be any spectrum. For X a finite CW-complex there is a natural identification $[\Sigma^\infty X, F]_r = \underset{n\to\infty}{colim}[\Sigma^{n+r} X, F_n]$

Before prove this lemma, we use it to prove:

[Prop] For X a finite CW-complex, there is a natural iso $[\Sigma^\infty X, H\mathbb{Z}]_{-r} \cong H^r(X;\mathbb{Z})$

pf:
$$\begin{aligned}[\Sigma^\infty X, H\mathbb{Z}]_{-r} &= \underset{n\to\infty}{colim}[\Sigma^{n-r} X, K(\mathbb{Z}, n)] \quad \text{(Lemma)} \\ &= \underset{n\to\infty}{colim}[\Sigma^{n} X, K(\mathbb{Z}, n+r)] \quad \text{[Rmk]} \\ &= \underset{n\to\infty}{colim}[X, K(\mathbb{Z}, r)] \overset{\text{similarly}}{=} H^r(X;\mathbb{Z})\end{aligned}$$

[Rmk] The reason for $\operatorname*{colim}_{n\to\infty}[\Sigma^{n-r}X, K(\mathbb{Z},n)] = \operatorname*{colim}_{n\to\infty}[\Sigma^{n}X, K(\mathbb{Z},n+r)]$ is quite subtle.

all based maps, not modulo homotopy

$$\mathrm{Map}(\Sigma^n X, K(\mathbb{Z},n+r)) \cong \mathrm{Map}(\Sigma^r\Sigma^{n-r}X, K(\mathbb{Z},n+r)) \cong \mathrm{Map}(\Sigma^{n-r}X, \Omega^r K(\mathbb{Z},n+r))$$

Choose connected component, i.e., modulo homotopy.

$\Omega K(\mathbb{Z},n+1) = K(\mathbb{Z},n)$ $\quad \cong \mathrm{Map}(\Sigma^{n-r}X, K(\mathbb{Z},n))$

Apply π_0, we have $[\Sigma^n X, K(\mathbb{Z},n+r)] \cong [\Sigma^{n-r}X, K(\mathbb{Z},n)]$. This iso compatible with colim maps, so $\operatorname*{colim}_{n\to\infty}[\Sigma^n X, K(\mathbb{Z},n+r)] = \operatorname*{colim}_{n\to\infty}[\Sigma^{n-r}X, K(\mathbb{Z},n)]$.

$$\begin{tikzcd}
{[\Sigma^n X, K(\mathbb{Z},n+r)]} \arrow[r] \arrow[d, "f_n"', "\cong"] & {[\Sigma^{n+1}X, K(\mathbb{Z},n+1+r)]} \arrow[d, "\cong"', "f_{n+1}"] \\
{[\Sigma^{n-r}X, K(\mathbb{Z},n)]} \arrow[r] & {[\Sigma^{n+1-r}X, K(\mathbb{Z},n+1)]}
\end{tikzcd}$$

(Check this commutes)

[Lemma] Let F be any spectrum. For X a finite CW-complex there is a natural identification $[\Sigma^\infty X, F]_r = \operatorname*{colim}_{n\to\infty}[\Sigma^{n+r}X, F_n]$,
where the colimit map is $[\Sigma^{n+r}X, F_n] \xrightarrow{\Sigma} [\Sigma^{n+r+1}X, \Sigma F_n] \xrightarrow{[-,\varepsilon_n]} [\Sigma^{n+r+1}X, F_{n+1}]$

Pf: **Idea:** Construct iso between $[\Sigma^\infty X, F]_r$ and $\operatorname*{colim}_{n\to\infty}[\Sigma^{n+r}X, F_n]$, since it's difficult to prove by universal prop.

(Note that n, r is fixed const)

Step1. Define $\varphi_n : [\Sigma^{n+r}X, F_n] \longrightarrow [\Sigma^\infty X, F]_r$
$[f^{n+r} : \Sigma^{n+r}X \to F_n] \longmapsto ?$

(Note that $[\Sigma^{n+r}X, F_n]$ are homotopy classes of maps between spaces while $[\Sigma^\infty X, F]_r$ are homotopy classes of pmaps)

Given $f^{n+r} : \Sigma^{n+r}X \to F_n$, we want to define a pmap from $\Sigma^\infty X$ to F. Firstly, we define $E' \subseteq \Sigma^\infty X$ be a cofinal subspectrum.

$E'_m := \begin{cases} \Sigma^m X & m \geq n+r \\ * & m < n+r \end{cases}$ with str map $\varepsilon_n \begin{cases} \text{str map of } \Sigma^\infty X & m \geq n+r \\ \text{unique based map} & m < n+r \end{cases}$

For any $K \subseteq (\Sigma^\infty X)_k$, $\Sigma^{n+r}(\Sigma^\infty X)_k \longrightarrow (\Sigma^\infty X)_{k+n+r} = E'_{k+n+r}$, hence $\Sigma^{n+r}K$ maps to E'_{k+n+r} leading to E' being a cofinal subspectrum.

Define a pmap $\varphi_n(f^{n+r}) : \Sigma^\infty X \to F$ of degree r defined on E' as following:
When $m \geq n+r$, $E'_m \to F_{m-r}$ is $\Sigma^m X \to F_{m-r}$ defined by composition

$$\Sigma^m X = \Sigma^{m-n-r}\Sigma^{n+r}X \xrightarrow{\Sigma^{m-n-r} f^{n+r}} \Sigma^{m-n-r}F_n \xrightarrow{\Sigma^{m-n-r-1}\varepsilon_n} \Sigma^{m-n-r-1}F_{n+1} \xrightarrow{\cdots} \cdots \to F_{m-r}$$

When $m < n+r$ $E'_m \to F_{m-r}$ is $* \to F_{m-r}$ is the unique based map.

Finally, we show that if $f^{n+r}, f'^{n+r} : \Sigma^{n+r}X \to F_n$ are homotopic, we want to show $\varphi_n(f^{n+r})$ homotopic to $\varphi_n(f'^{n+r})$.

Let $H : \Sigma^{n+r}X \times I \longrightarrow F_n$ be homotopy with $H(-,0) = f^{n+r}$, $H(-,1) = f'^{n+r}$.

We can rewrite homotopy $\bar{H}: \Sigma^{n+r}X \times I_+ \to F_n$ with $\bar{H}(-,0)=f^{n+r}$, $\bar{H}(-,1)=f'^{n+r}$, $\bar{H}(\Sigma X^{n+r} \times *) = *$. So $\bar{H}(\Sigma X^{n+r} \vee I_+) = \bar{H}(\Sigma X^{n+r} \times *) \cup \underbrace{\bar{H}(* \times I_+)}_{\text{base point maps to base point remain homotopy}} = * \cup * = *$

Thus we have homotopy $\bar{H}: \Sigma^{n+r}X \times I_+ / \Sigma^{n+r}X \vee I_+ = \Sigma^{n+r}X \wedge I_+ \to F_n$.

Define ψ_{m+r} be the composition $\psi_{m+r}: \Sigma^{m+r}X \wedge I_+ \xrightarrow{\Sigma^{m-n}H} \Sigma^{m-n}F_n \to F_m$, this is a degree r map $\psi: Cyl(\Sigma^\infty X) \to F$. Obviously $\psi i_0 = \varphi_n(f^{n+r})$, $\psi i_1 = \varphi_n(f'^{n+r})$.

$$\psi_{m+r}\, i_{0m+r}: \quad \begin{tikzcd} \{0\}_+ \wedge \Sigma^{m+r}X \arrow[r, "i_{0m+r}"] & I_+ \wedge \Sigma^{m+r}X \arrow[r, "\psi_{m+r}"] \arrow[dr, "\Sigma^{m-n}H"'] & F_m \\ & & \Sigma^{m-n}F_n \arrow[u] \end{tikzcd}$$

same   same

$$\varphi_n(f^{n+r})_m: \quad \Sigma^{m+r}X = \Sigma^{m-n}\Sigma^{n+r}X \xrightarrow{\Sigma^{m-n}f^{n+r}} \Sigma^{m-n}F_n \to F_m$$

So $\psi i_0 = \varphi_n(f^{n+r})$. Similarly, $\psi i_1 = \varphi_n(f'^{n+r})$. So $\varphi_n(f^{n+r}) \sim \varphi_n(f'^{n+r})$

<u>Step2</u> Define a map $\theta: \operatorname{colim}_{n\to\infty}[\Sigma^{n+r}X, F_n] \to [\Sigma^\infty X, F]_r$ and show it's well-defined.

We view $\operatorname{colim}_{n\to\infty}[\Sigma^{n+r}X, F_n]$ be $\bigcup_{n\in\mathbb{Z}}[\Sigma^{n+r}X, F_n]/\sim$ where $f^{n+r} \sim \underbrace{[-,\varepsilon_n]\circ\Sigma(f^{n+r})}$

$[\Sigma^{n+r}X, F_n] \to [\Sigma^{n+r+1}X, F_{n+1}]$

$$\begin{tikzcd} f^{n+r} \arrow[r, mapsto] & f^{n+r+1} \\ {[\Sigma^{n+r}X, F_n]} \arrow[r] \arrow[dr] & {[\Sigma^{n+r+1}X, F_{n+1}]} \arrow[d] \\ & \operatorname{colim}_{n\to\infty}[\Sigma^{n+r}X, F_n] \end{tikzcd}$$

Define $\theta: \operatorname{colim}_{n\to\infty}[\Sigma^{n+r}X, F_n] \to [\Sigma^\infty X, F]_r$

$$[f^{n+r}] \longmapsto \varphi_n(f^{n+r})$$

Let $f^{n+r+1} := [-,\varepsilon_n]\circ\Sigma(f^{n+r})$.

$\varphi_{n+1}(f^{n+r+1})$ is a pmap defined on cofinal subspectrum E'' with

$$E''_m = \begin{cases} \Sigma^m X & m \geq n+r+1 \\ * & m < n+r+1 \end{cases} . \qquad \varphi_{n+1}(f^{n+r+1}) = \begin{cases} \Sigma^m X \to F_{m-r} & m \geq n+r+1 \\ \text{unique based map} & m < n+r+1 \end{cases}$$

where $\Sigma^m X \to F_{m-r}$ is:

$$\Sigma^m X = \Sigma^{m-(n+1+r)}\Sigma^{n+1+r}X \xrightarrow{\Sigma^{m-n-1-r}f^{n+1+r}} \Sigma^{m-n-1-r}F_{n+1} \xrightarrow{\cdots} F_{m-r} .$$

Since

$$\begin{tikzcd} E' \arrow[dr, "\varphi_n(f^{n+r})"] & \\ & F \\ E'' \arrow[ur, "\varphi_{n+1}(f^{n+1+r})"'] & \end{tikzcd} , \quad \begin{cases} E'' \text{ is cofinal subspectrum of } E' \,\&\, E'' \\ \text{w.t.s. } \varphi_n(f^{n+r})|_{E''} = \varphi_{n+1}(f^{n+1+r})|_{E''} = \varphi_{n+1}(f^{n+1+r}) \end{cases}$$

$\Rightarrow$ $\varphi_n(f^{n+r})$ equals to $\varphi_{n+1}(f^{n+1+r})$ as pmaps.

$$\begin{array}{ccccccc}
\begin{tikzcd} \Sigma^m X \arrow[d, "\Sigma^{m-n-r} f^{n+r}"] \\ \Sigma^{m-n-r} F_n \arrow[d, "\Sigma^{m-n-r-1}\varepsilon_n"] \\ \Sigma^{m-n-1-r} F_{n+1} \arrow[d, "\vdots"] \\ F_{m-r} \end{tikzcd}
& \overset{\Sigma^{m-n-1-r}}{\longleftarrow} &
\begin{tikzcd} \Sigma^{n+1+r} X \arrow[d, "\Sigma f^{n+r}"'] \\ \Sigma F_n \arrow[d, "\varepsilon_n"] \\ F_{n+1} \end{tikzcd}
& \overset{\Sigma}{\longleftarrow} &
\begin{tikzcd} \Sigma^{n+r} X \arrow[d, "f^{n+r}"] \\ F_n \end{tikzcd}
& \overset{\Sigma^{m-n-r}}{\longrightarrow} &
\begin{tikzcd} \Sigma^m X \arrow[d, "\Sigma^{m-n-r} f^{n+r}"] \\ \Sigma^{m-n-r} F_n \arrow[d, "\vdots"] \\ F_{m-r} \end{tikzcd} \\
\varphi_{n+1}(f^{n+1+r}) & & f^{n+1+r} & & & & \varphi_n(f^{n+r})
\end{array}$$

we find red part are the same. so $\varphi_n(f^{n+r})$ equals $\varphi_{n+1}(f^{n+1+r})$ as a pmap.

<u>Step3</u> Show $\theta : \operatorname{colim}_{n\to\infty} [\Sigma^{n+r}X, F_n] \to [\Sigma^\infty X, F]_r$ is surj.

Let $g \in [\Sigma^\infty X, F]_r$ be a function of spectra of degree defined on cofinal subspectrum $K \subseteq \Sigma^\infty X$. (Note that we only pick a representable element in homotopy class.)

Idea: View $\operatorname{colim}_{n\to\infty}[\Sigma^{n+r}X, F_n]$ as a union of equiv class. To find preimage in it, it sufficese to find a represent element in $[\Sigma^{m+r}X, F_m]$ for some m.

Since X is a finite complex, $X \subseteq (\Sigma^\infty X)_0$ is a finite subcomplex. Since K is cofinal subspectrum of $\Sigma^\infty X$, $\exists m$ s.t. $\Sigma^m X \subseteq K_m$. Hence $g_m : K_m \to F_{m-r}$ has restriction $g_m|_{\Sigma^m X} : \Sigma^m X \to F_{m-r} \in [\Sigma^m X, F_{m-r}]$.

It suffices to show $\theta(g_m|_{\Sigma^m X}) = g$.

By definition, $\theta(g_m|_{\Sigma^m X})_k$ is :

$$\Sigma^k X = \Sigma^{k-m}\Sigma^m X \xrightarrow{\Sigma^{k-m} g_m|_{\Sigma^m X}} \Sigma^{k-m} F_{m-r} \xrightarrow{\cdots} F_{k-r}.$$

Since $g : \Sigma^\infty X \to F$ is a function of spectrum of degree r, we have

$$\begin{tikzcd} \Sigma^{k-m}\Sigma^m X \arrow[r, "id"] \arrow[d, "\Sigma^{k-m} g_m"'] & \Sigma^k X \arrow[d, "g_k"] \\ \Sigma^{k-m} F_{m-r} \arrow[r, "\cdots"] & F_{k-r} \end{tikzcd}$$

" $\llcorner$ " is $\theta(g_m|_{\Sigma^m X})_k$. So $g_k = \theta(g_m|_{\Sigma^m X})_k$

$$\Rightarrow g = \theta(g_m|_{\Sigma^m X})$$

<u>Step4</u> Show θ is inj. Idea: Replace X by $Cyl(X)$.

Suppose $\theta : \operatorname{colim}_{n\to\infty}[\Sigma^{n+r}X, F_n] \to [\Sigma^\infty X, F]_r$ with $\theta(f_0) \sim \theta(f_1)$.

$$f_0 \longmapsto \theta(f_0)$$
$$f_1 \longmapsto \theta(f_1)$$

So there exists $Cyl(\Sigma^\infty X) \xrightarrow{H} F$ with $H i_0 = \theta(f_0)$, $H i_1 = \theta(f_1)$.

By step3, we have $\theta' : \operatorname{colim}_{n\to\infty} [\Sigma^{n+r} Cyl(X), F_n] \to [\Sigma^\infty Cyl(X), F]_r$ is surj.

$$[\Sigma^\infty Cyl(X), F]_r = [Cyl(\Sigma^\infty X), F]_r \quad \Sigma^\infty Cyl(X) = Cyl(\Sigma^\infty X) \text{ by def}$$

So $\exists H' \in \operatorname{colim}_{n\to\infty} [\Sigma^{n+r} Cyl(X), F_n]$ s.t. $\theta'(H') = H$. Denote H'_m be the representative element in $[\Sigma^m Cyl(X), F_{m-r}]$. Same notation as $(f_0)_m$ and $(f_1)_m$.

$H'_m : \Sigma^m Cyl(X) \to F_{m-r}$ is homotopy between $(f_0)_m$ and $(f_1)_m$.

[Notation] Let CW denote the cat of CW-complexes and Ab denote the cat of ab grps.

[Def] A reduced cohomology theory on CW is a seq. of functors $e^n : CW \to Ab$ together with natural iso $e^n(X) \cong e^{n+1}\Sigma X$ s.t.

1. If $f, g : X \to Y$ are homotopic, then $e^n(f) = e^n(g) : e^n(X) \to e^n(Y)$
2. For each inclusion $A \hookrightarrow X$ in CW, the seq $e^n(X/A) \to e^n(X) \to e^n(A)$ is exact.
3. For $X = \vee_\alpha X_\alpha$ with inclusion $i_\alpha : X_\alpha \to X$, there exists a product map $\prod_\alpha i_\alpha : e^n(X) \to \prod_\alpha e^n(X_\alpha)$

[Thm] (Brown) Let e^n be a reduced cohomology theory on CW. Then there exists based spaces E_n and nat iso $e^n \Longrightarrow [-, E_n]$ so that the spaces E_n form an Ω-spectrum.

[Rmk] Note that: representability is natural and spaces E_n form an Ω-spectrum.

[Rmk] Show spaces E_n form an Ω-spectrum (ε_n' is a weak equiv)

$[X, E_n] \cong e^n(X) \cong e^{n+1}\Sigma X \cong [\Sigma X, E_{n+1}] \cong [X, \Omega E_{n+1}]$. By Yoneda lemma, there is an equiv $E_n \simeq \Omega E_{n+1}$.

Given $[X, A] \cong [X, B]$ for $\forall X$.

Then $A \cong [*, A] \cong \mathrm{Hom}([-,*], [-,A]) \cong \mathrm{Hom}([-,*], [-,B]) \cong [*, B] \cong B$.

[Rmk] Similar as the proof in $[\Sigma^\infty X, H\mathbb{Z}]_{-r} = H^r(X; \mathbb{Z})$, we have $[\Sigma^\infty X, E]_{-r} = e^r(X)$. Note that Σ^∞ is a functor sending space to a spectrum, this motivates following.

[Def] Let E be a spectrum. Define the generalized E-cohomology of degree r of a spectrum X to be $E^r X = [X, E]_{-r}$.

[Exp] 1. $E = \mathbb{S}$, the sphere spectrum. The generalized E-cohomology is stable cohomology: $\mathbb{S}^r X = [X, E]_{-r}$

2. Generalized MO-cohomology is called cobordism and generalized MU-cohomology is called complex cobordism.

§ Homotopy groups and weak equivalences

[Def] Let E be a spectrum. Define the homotopy grps of E by

$$\pi_r(E) := \operatorname*{colim}_{n\to\infty} \pi_{r+n} E_n$$

where maps in the colimit are $\pi_{r+n} E_n \longrightarrow \pi_{r+n+1} E_{n+1}$

$$[S^{r+n} \xrightarrow{f} E_n] \longmapsto [\Sigma S^{r+n} \xrightarrow{\Sigma f} \Sigma E_n \xrightarrow{\varepsilon_n} E_{n+1}]$$

[Fact] $\pi_r X \cong [\mathbb{S}, X]_r$

[Rmk] The spirit of cat theory is "morphisms are more important".
$\pi_r X \cong [\mathbb{S}, X]_r$, $E^r(X) = [X, E]_{-r}$, we represent grps by morphisms!

[Exp] 1. $\pi_n \mathbb{S} \cong [\mathbb{S}, \mathbb{S}]_n$ are stable homotopy grps of spheres.
2. $\pi_n MO \cong \Omega_n$ is the grp of compact oriented sm mf of dim n up to the bordism equiv.

[Thm] Let $E \to F$ be a function of spectra inducing an iso on π_*.
Then for any CW spectrum X, the map $[X, E]_r \to [X, F]_r$ is a bij. $\square$

Before prove this [Thm], let's see a [Coro] of it first.

[Coro] A pmap of CW spectra that induces an iso on π_* is an iso in the stable homotopy cat.

Pf: Idea: Reduce pmap to a function of spectra and use [Thm].

Let $f: E \to F$ be a pmap of CW spectra defined on cofinal subspectrum $E' \subseteq E$.
Then $f: E' \to F$ is a function of spectra. By [thm], $[X, E']_r \xrightarrow{f\circ -} [X, F]_r$ is a bij for any X.

① Take $X = F$, $[F, E']_r \xrightarrow{f\circ -} [F, F]_r$ is bij. Hence, for $[id] \in [F, F]_r$, $\exists [g] \in [F, E']_r$ s.t. $[fg] = [id]$. $[g]$ is represented by $g: F \to E'$, a pmap defined on cofinal $F' \subseteq F$.

② Take $X = E'$, $[E', E']_r \xrightarrow{f\circ -} [E', F]_r$ is bij. Consider $gf: E' \to E'$.

$[f\circ gf] = [(f\circ g) f] = [fg][f] = [f]$. Since $[f \circ id] = [f]$ and the inj of map, we obtain $[gf] = [id]$. Hence $\begin{cases} [f][g] = [id] \\ [g][f] = [id] \end{cases}$, meaning that $[f]$ is iso in stable homotopy cat.

[Rmk] Replace pmap with function is also true:

A function of CW spectra that induces an iso on π_* is an iso in the stable homotopy cat. Indeed, if function $f: E \to F$ induces iso on π_*, by [Thm] we have $[X,E]_r \xrightarrow{f_*} [X,F]_r$ is bij for $\forall X$. So by Yoneda's lemma, $E \xrightarrow{f} F$ is an equiv, which is an iso in stable homotopy cat. ($fg \sim id \Rightarrow [f][g] = [id]$)

□

The remain part will prove above [Thm]. We need to prove a lot of Lemmas and use the Adam's technique — induction on stable cells.

[Def] Let C_n denote the set of cells of CW complex E_n. The stable cells are $C = \operatorname{colim}_{n\to\infty} C_n$ where map in colimit is induced by $\varepsilon_n : \Sigma E_n \longrightarrow E_{n+1}$.

[Fact] E' is cofinal subspectrum of E iff the map of stable cells is bijection.

[Lemma] If $E \subseteq F$ is an inclusion of CW spectra and E is not cofinal. Then $\exists$ subspectrum J of F with $E \subseteq J \subseteq F$ and J contains exactly one more stable cell than E.

pf: Since E is not cofinal, $\exists$ a stable cell of F that doesn't contained in E. Say this cell has a representation $c \in F_n$ for some n. Since c is contained in finite subcomplex of F_n, there exist finite-subcomplexes of F_n containing stable cells not contained in E. We pick such finite subcomplex K with fewest number of cells. By minimality, K has only one cell that doesn't contained in E, denoted by c'. Write $K = L \cup c'$ where cells in L are stable cells contained in E. Since L is finite complex, $\exists\, m$ s.t. $\Sigma^m L \subseteq E_{m+n}$.

Let $J_i = E_i \cup \Sigma^{i-n} c'$ for $i \geq n+m$ and $J_i = E_i$ for $i < n+m$, this is the construction of J admits requirement.

§ Suspension is an equivalence

Let E be a CW spectrum.

[Def] The fake suspension $\Sigma_f E$ of E is the CW spectrum with spaces $(\Sigma E)_n = S^1 \wedge E_n$ and str maps $\Sigma(\Sigma E)_n = S^1 \wedge S^1 \wedge E_n \xrightarrow{1\wedge \varepsilon_n} S^1 \wedge E_{n+1} = (\Sigma E)_{n+1}$

[Def] Let $\Sigma^{-1}E$ be the CW spectrum with spaces $(\Sigma^{-1}E)_n = E_{n-1}$ and str maps $\Sigma(\Sigma^{-1}E)_n = \Sigma E_{n-1} \xrightarrow{\varepsilon_{n-1}} E_n = (\Sigma^{-1}E)_{n+1}$

[Prop] There are nat. iso. in the stable homotopy cat between

1. E and $\Sigma^{-1}\Sigma_f E$ 2. E and $\Sigma_f \Sigma^{-1} E$

[Rmk] This shows Σ_f can be inversed. The inverse of Σ_f is Σ^{-1}. If we show Σ equiv to Σ_f, then Σ has inverse Σ^{-1}.

Pf:

Step 1. Show $\Sigma^{-1}\Sigma_f E \cong \Sigma_f \Sigma^{-1} E$, and thus 1 and 2 in [prop] are equivalent.

$$\left.\begin{array}{l}(\Sigma^{-1}\Sigma_f E)_n = (\Sigma_f E)_{n-1} = S^1 \wedge E_{n-1} \\ (\Sigma_f \Sigma^{-1} E)_n = S^1 \wedge (\Sigma^{-1}E)_n = S^1 \wedge E_{n-1}\end{array}\right\} \Rightarrow (\Sigma^{-1}\Sigma_f E)_n = (\Sigma_f \Sigma^{-1} E)_n$$

Compute str map $\Sigma(\Sigma^{-1}\Sigma_f E)_n \to (\Sigma^{-1}\Sigma_f E)_{n+1}$: Let $X = \Sigma_f E$

$$\begin{array}{ccc}\Sigma(\Sigma^{-1}X)_n & \longrightarrow & (\Sigma^{-1}X)_{n+1} \\ \| & & \| \\ \Sigma X_{n-1} & \xrightarrow[\varepsilon'_{n-1}]{} & X_n\end{array}$$

where ε'_{n-1} is str map of spectrum X

$$\begin{array}{cccc}\varepsilon'_{n-1}: & \Sigma(\Sigma_f E)_{n-1} & \longrightarrow & (\Sigma_f E)_n \\ & \| & & \| \\ & S^1 \wedge S^1 \wedge E_{n-1} & \xrightarrow[1\wedge\varepsilon_{n-1}]{} & S^1 \wedge E_n\end{array}$$

$\rightsquigarrow$ So the nth str map of $\Sigma^{-1}\Sigma_f E$ is $1 \wedge \varepsilon_{n-1}$

Compute str map $\Sigma(\Sigma_f \Sigma^{-1}E)_n \longrightarrow (\Sigma_f \Sigma^{-1}E)_{n+1}$. Let $Y = \Sigma^{-1}E$

$$\begin{array}{ccc}\Sigma(\Sigma_f Y)_n & \longrightarrow & (\Sigma_f Y)_{n+1} \\ \| & & \\ S^1 \wedge S^1 \wedge Y_n & \xrightarrow[1\wedge\eta_n]{} & S^1 \wedge Y_{n+1}\end{array}$$

where η_n is str map of spectrum Y

$\eta_n : \Sigma(\Sigma^{-1}E)_n \longrightarrow (\Sigma^{-1}E)_{n+1}$ is ε_{n-1}. $\rightsquigarrow$ So the nth str map of $\Sigma_f\Sigma^{-1}E$ is $1 \wedge \varepsilon_{n-1}$.

Hence we have canonical iso $\Sigma^{-1}\Sigma_f E \cong \Sigma_f \Sigma^{-1} E$. Hence, it suffices to prove $\Sigma^{-1}\Sigma_f E \cong E$

> **Idea:** Recall [Thm]: If function of spectra inducing iso on π_*, then for CW spectrum, $[X, E]_r \to [X, F]_r$ is a bijection. This means E iso to F in stable homotopy cat. In our case, it suffices to construct a map $\Sigma^{-1}\Sigma_f E \to E$ induces iso on π_*.

Step 2. Construct $f : \Sigma^{-1}\Sigma_f E \to E$ and inverse of $\pi_r(f)$.

Let $f_n : (\Sigma^{-1}\Sigma_f E)_n = \Sigma E_{n-1} \to E_n$ defined to be ε_{n-1}, the str map of E. Then we have $\pi_r(f) : \pi_r \Sigma^{-1}\Sigma_f E \to \pi_r E$, where $\pi_r \Sigma^{-1}\Sigma_f E = \operatorname{colim}_{n\to\infty} \pi_{n+r}(\Sigma^{-1}\Sigma_f E)_n = \operatorname{colim}_{n\to\infty} \pi_{n+r}\Sigma E_{n-1}$ and $\pi_r E = \operatorname{colim}_{n\to\infty} \pi_{n+r} E_n$.

Suspension induces a map $\pi_{n+r-1}E_{n-1} \to \pi_{n+r}\Sigma E_{n-1}$.

i.e., suspension map $[S^{n+r-1}, E_{n-1}] \to [S^{n+r}, \Sigma E_{n-1}]$

This map induces a map between colimit as following:

$$\begin{tikzcd}
\pi_{n+r-1}E_{n-1} \arrow[rr] \arrow[dr] \arrow[dd] & & \pi_{n+r}E_n \arrow[dl] \arrow[dd] \\
& \operatorname{colim}_{n\to\infty} \pi_{n+r-1}E_{n-1} \arrow[dd, dotted, "\exists!\ \phi"] & \\
\pi_{n+r}\Sigma E_{n-1} \arrow[rr] \arrow[dr] & & \pi_{n+r+1}\Sigma E_n \arrow[dl] \\
& \operatorname{colim}_{n\to\infty} \pi_{n+r}\Sigma E_{n-1} &
\end{tikzcd}$$

There exists map

$$\phi_r : \operatorname{colim}_{n\to\infty} \pi_{n+r-1}E_{n-1} \to \operatorname{colim}_{n\to\infty} \pi_{n+r}\Sigma E_{n-1}$$

i.e., $\phi_r : \pi_r E \to \pi_r \Sigma^{-1}\Sigma_f E$

Step 3 Show ϕ and $\pi_r(f)$ are inverse.

[Fact] Given a diagram

$$\begin{tikzcd}
\cdots & A_n \arrow[dr] & & A_{n+1} & \cdots \\
B_n \arrow[ur] & & B_{n+1} \arrow[ur] & &
\end{tikzcd}$$

The upward and downward arrows induce natural inverse iso $\operatorname{colim}_{n\to\infty} A_n \cong \operatorname{colim}_{n\to\infty} B_n$

In our case we have $A_n = \pi_{n+r}E_n$ and $B_n = \pi_{n+r}\Sigma E_{n-1}$, and arrows

$$\begin{tikzcd}
\cdots & A_n \arrow[dr, "\phi_r"] & & A_{n+1} & \cdots \\
B_n \arrow[ur, "\pi_r(f)"] & & B_{n+1} \arrow[ur] & &
\end{tikzcd}$$

. By [Fact], $\pi_r(f)$ and ϕ_r induce natural inverse iso

$$\pi_r E \cong \pi_r \Sigma^{-1}\Sigma_f E$$

[Def] Define spectrum ΣE whose nth space is $S^1 \wedge E_n$ and str maps

$$S^1 \wedge S^1 \wedge E_n \xrightarrow{\tau\wedge 1} S^1 \wedge S^1 \wedge E_n \xrightarrow{1\wedge \varepsilon_n} S^1 \wedge E_{n+1} = (\Sigma E)_{n+1}$$

where $\tau : S^1 \wedge S^1 \to S^1 \wedge S^1$ is the swap.

[prop] There is nat iso in the stable homotopy cat between $\Sigma_f E$ and ΣE.

[Rmk] Whether or not swap S^1 doesn't imapact definition.

Before prove this prop, we see some corollary first.

[Coro] (Replace Σ_f by Σ in above prop) There are nat. iso. in the stable homotopy cat between 1. E and $\Sigma^{-1}\Sigma E$ 2. E and $\Sigma\Sigma^{-1}E$

[Coro] Suspension $\Sigma : [E,F]_* \longrightarrow [\Sigma E, \Sigma F]_*$ induces a bijection.

pf: We have bij $[\Sigma^{-1}\Sigma E, \Sigma^{-1}\Sigma F]_* \cong [E,F]$, since $\Sigma^{-1}\Sigma E \cong E$, $\Sigma^{-1}\Sigma F \cong F$.

Define a map $\Sigma^{-1} : [X,Y] \longrightarrow [\Sigma^{-1}X, \Sigma^{-1}Y]$
$f_n \longmapsto f'_n = f_{n-1}$

$$\begin{tikzcd}
f_n \arrow[rr, mapsto] \arrow[ddd, mapsto] & & 1\wedge f_n \arrow[ddd, mapsto] \\
{[E,F]_*} \arrow[r, "\Sigma"] \arrow[d, "\cong"] & {[\Sigma E, \Sigma F]_*} \arrow[dl, "\Sigma^{-1}"] & \\
{[\Sigma^{-1}\Sigma E, \Sigma^{-1}\Sigma F]_*} & & \\
\boxed{\eta_n = 1\wedge f_{n-1}} & = & \theta_n = 1\wedge f_{n-1}
\end{tikzcd}$$

How to compute η_n:
The iso between E and $\Sigma^{-1}\Sigma E$ are
$\varepsilon^E_{n-1} : (\Sigma^{-1}\Sigma E)_n = \Sigma E_{n-1} \longrightarrow E_n$
similarly $\varepsilon^F_{n-1} : (\Sigma^{-1}\Sigma F)_n \longrightarrow F_n$ is an iso.

So
$$\begin{tikzcd}
E_n \arrow[r, "f_n"] & F_n \\
(\Sigma^{-1}\Sigma E)_n \arrow[r, "\eta_n"] \arrow[u, "\varepsilon^E_{n-1}", "\cong"'] & (\Sigma^{-1}\Sigma F)_n \arrow[u, "\varepsilon^F_{n-1}", "\cong"']
\end{tikzcd}$$
. By commutative diagram of function of spectra, $\eta_n = \Sigma f_{n-1} = 1\wedge f_{n-1}$.

Hence the diagram commutes, leading to Σ^{-1} is inverse to Σ. $\square$

Finally comes the proof for [prop]:

[prop] There is nat iso in the stable homotopy cat between $\Sigma_f E$ and ΣE.

To prove this, we use S^2-spectrum.

[Def] An S^2-spectrum is a seq of spaces X_n with n in $\mathbb{Z}$ together with str maps $S^2\wedge X_n \to X_{n+1}$.

[Rmk] S^2-spectrum is NOT a spectrum. We can call previous spectrum as S^1-spectrum.

[Construction] 1. Every spectrum E determines a S^2-spectrum $R(E)$.

$R(E)_n = E_{2n}$ with str. map $S^2\wedge R(E)_n = S^2\wedge E_{2n} \xrightarrow{\cong} S^1\wedge S^1\wedge E_{2n} \xrightarrow{1\wedge \varepsilon_{2n}} S^1\wedge E_{2n+1} \xrightarrow{\varepsilon_{2n+1}} E_{2n+2} = R(E)_{n+1}$

2. Every S^2-spectrum X determines a spectrum $L(X)$.

$$\begin{cases} L(X)_{2k} = X_k \\ L(X)_{2k+1} = S^1\wedge X_k \end{cases}$$ with str map $$\begin{cases} S^1\wedge L(X)_{2k} = S^1\wedge X_k \xrightarrow{id} S^1\wedge X_k = L(X)_{2k+1} \\ S^1\wedge L(X)_{2k+1} = S^1\wedge S^1\wedge X_k \cong S^2\wedge X_k \longrightarrow X_{k+1} \end{cases}$$
str map of S^2 spectrum X

3. $LR(E)$ is a spectrum with

$$LR(E)_{2k} = \begin{cases} R(E)_k = E_{2k} \\ S^1\wedge R(E)_k = S^1\wedge E_{2k} \end{cases}$$ with str map $$\begin{cases} \underset{S^1\wedge E_{2k}}{S^1\wedge LR(E)_{2k}} \xrightarrow{id} \underset{S^1\wedge E_{2k}}{S^1\wedge LR(E)_{2k+1}} \\ \underset{S^1\wedge LR(E)_{2k+1}}{S^1\wedge S^1\wedge E_{2k}} \xrightarrow{1\wedge\varepsilon_{2k}} S^1\wedge E_{2k+1} \xrightarrow{\varepsilon_{2k+1}} E_{2k+2} = LR(E)_{2k+2} \end{cases}$$

4. $LR(E) \longrightarrow E$ is a function which is identity in evenly indexed levels and the str map $S^1\wedge E_{2n} \longrightarrow E_{2n+1}$ in odd levels.

Claim: $LR(E) \to E$ is an iso in stable homotopy cat.

Pf of the claim: It suffices to show $LR(E) \to E$ induces iso on π_*.

Note that given diagram
$$\begin{tikzcd} \cdots A_n \arrow[r] \arrow[d, "\wr\|"'] & A_{n+1} \arrow[d, "\wr\|"] \cdots \\ \cdots B_n \arrow[r] & B_{n+1} \cdots \end{tikzcd}$$
induces iso $\operatorname{colim}_{n\to\infty} A_n \cong \operatorname{colim}_{n\to\infty} B_n$

(easily proved by univ prop)

Hence, to show $\pi_r LR(E) \to \pi_r E$ is iso, it suffices to show $\pi_{n+r} LR(E)_n \xrightarrow{\cong} \pi_{n+r} E_n$ at each degree. For $n=2k$, it's $\pi_{n+r}(id)=id$, which is an iso. For $n=2k+1$, it's $\pi_{n+r}(\varepsilon_{2k})$, which is also an iso. $\square$

str map is weak homotopy equiv, induces iso on π_*.

proof for the [prop]:

[Fact] If $R(\Sigma_f E)$ and $R(\Sigma E)$ are iso in the cat of S^2-spectrum, then $\Sigma_f E$ and ΣE are iso in stable homotopy theory.

maps are homotopy classes of p maps

$$\left.\begin{aligned} R(\Sigma_f E)_n &= (\Sigma_f E)_{2n} = S^1 \wedge E_{2n} \\ R(\Sigma E)_n &= (\Sigma E)_{2n} = S^1 \wedge E_{2n} \end{aligned}\right\} \Rightarrow R(\Sigma_f E)_n = R(\Sigma E)_n .$$

str map of $R(\Sigma_f E)$: $S^2 \wedge R(\Sigma_f E)_n \longrightarrow R(\Sigma_f E)_{n+1}$

$$S^2\wedge(\Sigma_f E)_{2n} = S^2 \wedge (S^1 \wedge E_{2n}) \cong S^1\wedge S^1 \wedge (S^1\wedge E_{2n}) \xrightarrow{1\wedge 1\wedge \varepsilon_{2n}} S^1\wedge S^1 \wedge E_{2n+1} \xrightarrow{1\wedge \varepsilon_{2n+1}} S^1 \wedge E_{2n+2}$$

str map of $R(\Sigma E)$: $S^2 \wedge R(\Sigma E)_n \to R(\Sigma E)_{n+1}$ is:

$$S^2\wedge R(\Sigma E)_n = S^2 \wedge (\Sigma E)_{2n} \cong S^1\wedge S^1\wedge (S^1\wedge E_{2n}) \xrightarrow{1\wedge\tau\wedge 1} S^1\wedge S^1\wedge S^1 \wedge E_{2n} \xrightarrow{1\wedge 1\wedge \varepsilon_{2n}} S^1\wedge S^1\wedge E_{2n+1}$$
$$\xrightarrow{\tau\wedge 1} S^1\wedge S^1\wedge E_{2n+1} \xrightarrow{1\wedge \varepsilon_{2n+1}} S^1\wedge E_{2n+2} = R(\Sigma E)_{n+1}$$

Hence, str maps are differ by the composition of TWO swaps on $S^1\wedge S^1 \wedge S^1$.

The composition of two swaps has the degree $(-1)\times(-1)=1$, so there is a homotopy between two str maps. Define S^2-spectrum H whose nth space is $R(\Sigma E)_n \wedge [0,1]_*$ and str maps $S^2 \wedge R(\Sigma E)_n \wedge [0,1]_* \xrightarrow{\text{homotopy}} (R\Sigma E)_{n+1} \hookrightarrow (R\Sigma E)_{n+1} \wedge * \hookrightarrow (R\Sigma E)_{n+1} \wedge [0,1]_*$ where the homotopy "$S^2\wedge R(\Sigma E)_n \wedge [0,1]_* \xrightarrow{\text{homotopy}} (R\Sigma E)_{n+1}$" is the homotopy between str maps of $R(\Sigma_f E)$ and $R(\Sigma E)$.

Let $\varphi_n: R(\Sigma_f E)_n \cong R(\Sigma_f E)_n \wedge \{0\}_* \hookrightarrow R(\Sigma_f E)_n \wedge [0,1]_* = H_n$

$\psi_n: R(\Sigma E)_n \cong R(\Sigma E)_n \wedge \{1\}_* \hookrightarrow R(\Sigma E)_n \wedge [0,1]_* = H_n$ $\quad (R(\Sigma_f E)_n = R(\Sigma E)_n)$

Obviously, we have comm. diagram:

$$\begin{tikzcd} S^2\wedge R(\Sigma_f E)_n \arrow[r, "\text{str map}"] \arrow[d, "1\wedge\varphi_n"'] & R(\Sigma_f E)_{n+1} \arrow[d, "\varphi_{n+1}"] \\ S^2\wedge H_n \arrow[r, "\text{str map}"] & H_{n+1} \end{tikzcd} \qquad \begin{tikzcd} S^2\wedge R(\Sigma E)_n \arrow[r, "\text{str map}"] \arrow[d, "1\wedge\psi_n"'] & R(\Sigma E)_{n+1} \arrow[d] \\ S^2\wedge H_n \arrow[r, "\text{str map}"] & H_{n+1} \end{tikzcd},$$

So φ and ψ are function of S^2-spectra.

$\pi_{n+r}(R\Sigma_f E)_n \longrightarrow \pi_{n+r} H_n$ is induced by $(R\Sigma_f E)_n \wedge \{0\}_* \hookrightarrow R(\Sigma_f E)_n \wedge [0,1]_*$, which is homotopy equiv since $[0,1]$ is contractible. So $\pi_{n+r}(R\Sigma_f E)_n \longrightarrow \pi_{n+r} H_n$ is an iso. Hence $\pi_r(R\Sigma_f E) = \operatorname{colim}_{n\to\infty} \pi_{n+r}(R\Sigma_f E)_n \cong \operatorname{colim}_{n\to\infty} \pi_{n+r} H_n = \pi_r H$.

Similarly, $\pi_r(H) \cong \pi_r(R\Sigma E)$. So we have a map $R\Sigma_f E \to R\Sigma E$ induces iso $\pi_r(R\Sigma E) \cong \pi_r(R\Sigma_f E)$, leading to $\Sigma_f E \cong \Sigma E$ in stable homotopy cat.

§ Smash products and internal function objects in the stable homotopy cat

[Fact] In stable homotopy cat, $F(X\wedge Y, Z) = F(X, F(Y,Z))$ where X, Y, Z are objects in stable homotopy cat and $F(X,Y) = \mathrm{Map}_*(X,Y)$ (NOT modulo homotopy).

[Rmk] $F(X,Y)$ is called internal function object.

[prop] $\Sigma^\infty A \wedge \Sigma^\infty B = \Sigma^\infty (A\wedge B)$.

pf: $(\Sigma^\infty A\wedge \Sigma^\infty B)_n = \Sigma^n A \wedge \Sigma^n B = \Sigma^n (A\wedge B) = (\Sigma^\infty (A\wedge B))_n$

[Def] Let W be a CW complex and X be a spectrum. Define $X\wedge W$ be a spectrum whose nth space is $(X\wedge W)_n = X_n \wedge W$ and str map $\varepsilon_n \wedge 1 : \Sigma X_n \wedge W \to X_{n+1}\wedge W$.

[Rmk] If X is an obj in stable homotopy cat, we can also define $X\wedge W$ by taking any spectrum representing X.

[Rmk] $W \mapsto [X\wedge W, Y]$ is a generalized cohomology theory. The corresponding object in stable homotopy cat is $F(X,Y)$.

§ Cofiber sequences are the same as fiber sequences.

Similarly as Top_*, there are cofiber sequences in stable homotopy cat.

Recall: For $X, Y \in \mathrm{Top}_*$, $f: X\to Y$ is a conti. map. Reduced mapping cone $Y\cup_f CX := (Y \amalg (X\wedge I))/\sim$ where $(x,1)\sim f(x)$. Very f by a homotopy, $Y\cup_f CX$ changes by a homotopy equiv.

[Construction] (Reduced mapping cone of a map in stable homotopy cat, $Y\cup_f CX$)
Let $f: X\to Y$ be a mor in stable homotopy cat. Say f is represented by a function of spectra $f: X' \to Y$ where X' is a cofinal subspectrum of X. We can always replace representing element of homotopy class (replace f by a homotopic map) s.t. $f'_n : X'_n \to Y_n$ is a cellular map.
Define $Y\cup_f CX$ be the spectrum whose nth space is $Y_n \cup_{f_n} CX'_n$ and whose str maps are $\Sigma Y_n \cup_{f_n} CX'_n = \Sigma Y_n \amalg \Sigma X'_n \wedge I/\sim \;\longrightarrow Y_{n+1} \amalg X_{n+1}\wedge I/\sim$ induced by str maps of X & Y.
Very f by a homotopy doesn't change iso class of $Y\cup_f CX$, hence it's well defined.

[Construction] (spectrum X/A) Let $i: A\to X$ be inclusion of a closed subspectrum.
Define X/A whose nth space is X_n/A_n and whose str maps are
$\Sigma(X_n/A_n) = \Sigma X_n/\Sigma A_n \to X_{n+1}/A_{n+1}$ induced by str map $\Sigma X_n \to X_{n+1}$ since ΣA_n maps to A_{n+1} under the str map of X.
Since $X_n \cup_i CA_n \to X_n/A_n$ is homotopy equiv, $X\cup_i CA \to X/A$ induces iso on π_*, leading to $X\cup_i CA \cong X/A$ in stable homotopy equiv.

[Def] A cofiber seq is any seq equiv to a seq of the form $X \xrightarrow{f} Y \xrightarrow{i} Y\cup_f CX$

[prop] For each Z, the seq $[Y\cup_f CX, Z] \to [Y,Z] \to [X,Z]$ is exact.

pf: Denote $X \xrightarrow{f} Y \xrightarrow{f_2} Y\cup_f CX$, then
$$\begin{array}{ccccc} [Y\cup_f CX, Z] & \to & [Y,Z] & \to & [X,Z] \\ k & \longmapsto & kf_2 & \longmapsto & kf_2 f \end{array}$$
Since $X \xrightarrow{f_2 f} Y\cup_f CX$ is null, so $kf_2 f$ is null, leading to "Im $\subseteq$ ker".

To prove "ker $\subseteq$ Im", let $g: Y \to Z$ defined on cofinal subspectrum Y' s.t. gf is null in $[X, Z]$. Say gf is defined as function on X' and there is homotopy $H: X' \wedge [0,1]_+ \to Z$ giving a homotopy between gf and const map. Let Y'' be a cofinal subspectrum of Y that containing $f(X')$, then we have reduced mapping cone $Y'' \cup_f CX' = Y'' \amalg (X' \wedge [0,1]_+)/_{(x,1)\sim f(x)}$

Since $H(x,1) = g' f(x)$, so $H: X' \wedge [0,1]_+ \longrightarrow Z$ and $g: Y'' \to Z$ induces a map $g \amalg H: Y'' \cup_f CX' \longrightarrow Z$. w.t.s $(g \amalg H) f_2 = g$

[Fact] $(Y \cup_f CX) \cup_i CY \cong \Sigma X$

[Construction] (Any map can be extended to a cofiber seq, part I)

Let $f: X \to Y$ be a mor in stable homotopy cat. We can extend $X \xrightarrow{f} Y$ to the right as following: $X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX \longrightarrow \underset{\Sigma X}{\overset{\cong}{(Y \cup_f CX) \cup_i CY}} \to \Sigma Y \to \Sigma(Y \cup_f CX) \to \Sigma\Sigma X \to \cdots$ (1)

all three term seq are cofiber seq.

We've known $X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX \to \Sigma X$ s.t. all three term seq are cofiber seq. Let $X = Y$, $Y = Y \cup_f CX$, this term can be ΣY

[prop] Desuspension Σ^{-1} and suspension Σ preserve cofiber seq.

pf: Let $X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX$ be a cofiber seq. Apply Σ to it, we have $\Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma i} \Sigma(Y \cup_f CX) = \Sigma Y \cup_f C\Sigma X$, so it's a cofiber seq. Similar proof for Σ^{-1}.

[Construction] (Any map can be extended to a cofiber seq, part II)

Since desuspension Σ^{-1} preserves cofiber seq, we apply Σ^{-1} to (1)

$$\Sigma^{-1} X \xrightarrow{\Sigma^{-1} f} \Sigma^{-1} Y \longrightarrow \Sigma^{-1}(Y \cup_f CX) \longrightarrow \underbrace{\underset{X}{\underset{\|}{\Sigma^{-1}\Sigma X}} \longrightarrow \underset{Y}{\underset{\|}{\Sigma^{-1}\Sigma Y}} \to \Sigma^{-1}\Sigma(Y \cup_f CX) \to \Sigma^{-1}\Sigma\Sigma X \to \cdots}_{\text{This part is seq (1)}}$$

slogan: Σ^{-1} is the inverse of Σ in stable homotopy cat

Apply Σ^{-1} to (1) extend (1) to the left. Similarly, apply Σ^{-n} to (1) can extend (1) start from $\Sigma^{-n} X \to \Sigma^{-n} Y \to \Sigma^{-n}(Y \cup_f CX) \to \Sigma^{-n+1} X \to \cdots$

[Coro] Let $X \to Y \to Z$ be a cofiber seq in the stable homotopy cat, then for any W, the seq $\cdots \to [X, W]_{n+1} \to [Z, W]_n \to [Y, W]_n \to [X, W]_n \to \cdots$ is exact.

pf: By above construction, we have seq s.t. all three terms are cofiber seqs

$$\cdots \to \Sigma^n X \to \Sigma^n Y \to \Sigma^n Z \to \Sigma^{n+1} X \to \cdots \quad (2)$$

Apply $[-, W]$ we obtain

$$\cdots \to [\Sigma^{n+1} X, W] \to [\Sigma^n Z, W] \to [\Sigma^n Y, W] \to [\Sigma^n X, W] \to \cdots \quad (3)$$

Since for any W, cofiber seq induces exact $[Y \cup_f CX, W] \to [Y, W] \to [X, W]$, we have seq (3) is exact. (Since any three terms in (2) is cofiber seq)

Since $[\Sigma^n Y, W] = [Y, W]_n$, we rewrite exact seq (3)

$$\cdots \to [X, W]_{n+1} \to [Z, W]_n \to [Y, W]_n \to [X, W]_n \to \cdots$$

which is what we require.

[Rmk] Why $[\Sigma^n Y, W] = [Y, W]_n$? $(\Sigma^{-n} Y)_m = Y_{n-m}$, Σ^n is inverse of Σ^{-n}, so $(\Sigma^n Y)_m \cong Y_{n+m}$. Then $[\Sigma^n Y, W] = \{ f_m : (\Sigma^n Y)_m = Y_{n+m} \to W_m \} = [Y, W]_n$

[prop] Let $X \xrightarrow{f} Y \xrightarrow{i} Z$ be a cofiber seq in stable homotopy cat. For any W, the seq $\cdots \to [W, X]_n \to [W, Y]_n \to [W, Z]_n \to [W, X]_{n-1} \to \cdots$ is exact.

Pf: The seq is equiv to $\cdots \to [W, \Sigma^{-n} X] \to [W, \Sigma^{-n} Y] \to [W, \Sigma^{-n} Z] \to \cdots$
which is induced by $\cdots \to \Sigma^{-n} X \to \Sigma^{-n} Y \to \Sigma^{-n} Z \to \Sigma^{-n+1} X \to \cdots$.
all three terms are cofiber seq's

Hence it suffices to show $[W, X] \xrightarrow{f\circ -} [W, Y] \xrightarrow{i\circ -} [W, Z]$ is exact.
Since the composition $X \to Z$ is null, we have $[W, X] \to [W, Z]$ is zero map.
Then we w.t.s. any pmap $g: W \to Y \in [W, Y]$ s.t. ig is null is a image of $f\circ -$.
Since ig is null, there exists a homotopy $h: W \times I \to Z$ with $h(-, 0) = gi$ and $h(-, 1) = *$.
So h induces map $h: W \times I / W \times 1 \to Z$, i.e., $h: CW \to Z$, with $CW = W \times I / W \times 1$.

Consider the following comm diagram:

$$\begin{array}{ccccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{i} & Z & \xrightarrow{k} & \Sigma X & \longrightarrow & \Sigma Y \\
 & & \uparrow g & & \uparrow h & & \uparrow j & & \uparrow \Sigma g \\
W & \xrightarrow{id} & W & \xrightarrow{\iota} & CW & \longrightarrow & \Sigma W & \longrightarrow & \Sigma W \\
 & & \cong & & & & & & \\
 & & W \times \{0\} & & & & & &
\end{array}$$

where j is constructed as following: $kh: CW \to \Sigma X$ restricts on $W \times \{0\}$ is $kig = 0$ since $ki = 0$. so kh induces a map $j := kh: \underbrace{CW / W\times\{0\} \cup * \times I}_{\Sigma W} \to \Sigma X$
i.e., $j: \Sigma W \to \Sigma X$

Since Σ is an equiv, we have $\Sigma^{-1} j : W \to X$. $f \circ \Sigma^{-1} j = g$, we find the preimage of g. □

[Rmk] There is an equivalent definition of cofiber seq's. A seq $X \to Y \to Z$ is a cofiber seq if $X \to Z$ is null and $\cdots \to [X, W]_{n+1} \to [Z, W]_n \to [Y, W]_n \to [X, W]_n \to \cdots$ is exact. Dually, a fiber seq can be defined as seq $X \to Y \to Z$ s.t. $X \to Z$ is null map and $\cdots \to [W, X]_n \to [W, Y]_n \to [W, Z]_n \to \cdots$ is exact. Therefore, the prop that cofiber seq both satisfying two seq's are exact leading to the statement: cofiber seq and fiber seq are the same in the stable homotopy cat.

§ Spanier – Whitehead Duality

- Notation: We always denote $\Sigma^\infty X$ as its base space X. e.g., S^0 also denotes $\Sigma^\infty S^0 =: \mathbb{S}$ the sphere spectrum.

- Although we haven't constructed, we assume that for any objects X, Y in stable homotopy cat, there exists a smash product $X \wedge Y$ satisfying:

① $\exists$ spectrum $F(X, Y)$ s.t. $[W \wedge X, Y] = [W, F(X, Y)]$, for $\forall W \in$ stable homotopy cat.
called function spectrum

② $X \wedge S^0$ and $S^0 \wedge X$ are canonically iso to X. ($(X\wedge \mathbb{S})_n = X_n \wedge S^n \cong \Sigma X_0 \wedge \Sigma S^0 \cong \Sigma^n(X_0 \wedge S^0) \cong \Sigma^n X_0 \cong X_n$)
sphere spectrum $\mathbb{S}$

③ smash products is associative and commutative, i.e., for $\forall X, Y, Z$ $\exists$ canonical iso $X\wedge Y \cong Y \wedge X$, $(X\wedge Y)\wedge Z \cong X \wedge (Y \wedge Z)$.

④ smash product $\wedge$ preserves cofiber sequences.

[Def] Let X be an obj in stable homotopy cat. A dual of X is an obj Y equipped with maps $e: X\wedge Y \to S^0$, $\eta: S^0 \to Y \wedge X$ s.t. the compositions:

$$X\wedge S^0 \xrightarrow{1\wedge\eta} X\wedge Y \wedge X \xrightarrow{e\wedge 1} S^0 \wedge X$$
$$S^0 \wedge Y \xrightarrow{\eta\wedge 1} Y\wedge X\wedge Y \xrightarrow{1\wedge e} Y \wedge S^0$$

are the canonical iso.

[Rmk] 1. This canonical iso's are sometimes called "the identity".
2. This definition make sense in any monoidal cat, which admits an associative and commutative multiplication.
3. We find this definition quite similar to definition of adjoint function. Actually, the adjoint functors from a cat to itself is the same data as dual objs in the cat of functors from a cat to itself.
4. Very similar to the proof of adjoint functors, we have: if Y is a dual of X, then X is a dual of Y.

[prop] Duals are unique. More explicitly, the dual of X if exists, is $F(X, S^0)$.
pf:
[Fact] If Y is dual to X, then for $\forall W, Z$, the map
$[Z\wedge Y, W] \xrightarrow{[-\wedge X, -\wedge X]} [Z\wedge Y \wedge X, W\wedge X] \xrightarrow{[-\wedge\eta, -]} [Z, W\wedge X]$ is an iso.
If Y is dual of X, then X is dual of Y. Switch X and Y in the [Fact], we have $[Z\wedge X, W] \to [Z\wedge X \wedge Y, W\wedge Y] \to [Z, W\wedge Y]$ is an iso.
Let $W = S^0$ in above maps, we obtain $[Z\wedge X, S^0] \cong [Z, Y]$, $\forall Z$.
Then $[Z, F(X, S^0)] \cong [Z\wedge X, S^0] \cong [Z, Y]$. By Yoneda lemma, we have $Y \cong F(X, S^0)$. □

Since dual is unique, we write DX for the dual of X. DX is called the Spanier-Whitehead dual of X.

[Rmk] $D^2X \cong X$.

[prop] D is a contravariant functor on the subcat of dualizable objects.

Pf: Let X_1, X_2 be dualizable objs and $f: X_1 \to X_2$ be a mor in the stable homotopy cat. Then $[DX_2, DX_1] = [DX_2, S^0\wedge DX_1] \cong [DX_2 \wedge X_1, S^0]$
$[Z\wedge X, W] \cong [Z, W\wedge DX]$

$$= [X_1 \wedge DX_2, S^0] \underset{[X\wedge Y, W] = [X, F(X,W)]}{=} [X_1, F(DX_2, S^0)] \underset{D(DX_2) = F(DX_2, S^0)}{=} [X_1, D(DX_2)] \underset{D^2X_2 = X_2}{=} [X_1, X_2]$$

Hence, $\exists f' \in [DX_2, DX_1]$ corresponds to $f \in [X_1, X_2]$. Every step preserves composition, so D is a contravariant functor.

[Def] A CW spectrum is finite if it has a finite number of stable cells.

The goal of the rest part is to show:

[Thm] Let X be a finite CW spectrum. The dual of X exists.

[Construction] Let $A \subset X$ be an inclusion of CW spectra. The map $A \to X$ induces a map of cohomology theory $[X\wedge(-), S^0] \to [A\wedge(-), S^0]$ as following:
$F(X_1, X_2)$ is natural for X_1, X_2. So $A \to X$ induces a map $F(X, S^0) \xrightarrow{f} F(A, S^0)$.

$$[X\wedge(-), S^0] = [-, F(X, S^0)] \xrightarrow{f\circ -} [-, F(A, S^0)] = [A\wedge(-), S^0]$$

[Lemma] $F(X/A, S^0) \cong \Sigma^{-1}F(A, S^0)/F(X, S^0)$, i.e., $\Sigma D(X/A) \cong DA/DX$

Pf: By Yoneda Lemma, it suffices to show $[W, F(X/A, S^0)] \cong [W, \Sigma^{-1}F(A,S^0)/F(X,S^0)], \forall W$.

$F(X,S^0) \to F(A,S^0) \to F(A,S^0)/F(X,S^0)$ is a fiber seq, we extend it to left by Σ^{-1} and just pick three terms, which is a fiber seq:

$$\Sigma^{-1}F(A,S^0)/F(X,S^0) \xrightarrow{g} F(X,S^0) \to F(A,S^0)$$

(Similarly to universal prop of kernel, fiber $K \to X \xrightarrow{f} Y$ has universal prop that for any $W \to X \xrightarrow{f} Y$ being null, $\exists!$ map $W \to K$ rendering diagram commutes)

Pick map $F(X/A, S^0) \to F(X, S^0)$ whose image in $F(A, S^0)$ is null,

$$\begin{tikzcd}
\Sigma^{-1}F(A,S^0)/F(X,S^0) \arrow[r, "g"] & F(X,S^0) \arrow[r] & F(A,S^0) \\
F(X/A, S^0) \arrow[u, dashed, "\exists! f"] \arrow[ur, "h"'] & &
\end{tikzcd}$$

$\exists! f$ s.t. diagram commutes.

$A \to X \to X/A \to \Sigma A \to \Sigma X \to \cdots$ is a fiber seq for each 3 terms.

Since $\wedge$ preserves cofiber seq (same as fiber seq), we have fiber seq

$\cdots \to W\wedge\Sigma X \to W\wedge\Sigma A \to W\wedge X/A \to W\wedge X \to W\wedge A \to \cdots$ it induces exact seq:

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & {[W, F(\Sigma X, S^0)]} \arrow[r] \arrow[d, phantom, "\overset{\textcircled{1}}{\shortparallel}"] & {[W, F(\Sigma A, S^0)]} \arrow[r] \arrow[d, phantom, "\shortparallel"] & {[W, F(X/A, S^0)]} \arrow[r] & {[W, F(X,S^0)]} \arrow[r] & {[W, F(A,S^0)]} \arrow[r] & \cdots \\
& {[W, \Sigma^{-1}F(X,S^0)]} & {[W, \Sigma^{-1}F(A,S^0)]} & & & &
\end{tikzcd}$$

① $[W, \Sigma^{-1}F(X,S^0)] = [\Sigma W, F(X,S^0)]$
$= [\Sigma W\wedge X, S^0] = [W\wedge\Sigma X, S^0]$
$= [W, F(\Sigma X, S^0)]$

By map $f: F(X/A) \to \Sigma^{-1}F(A,S^0)/F(X,S^0)$, we have

$$\begin{tikzcd}[column sep=small]
{[W, \Sigma^{-1}F(X,S^0)]} \arrow[r] \arrow[d, "id_*"] & {[W, \Sigma^{-1}F(A,S^0)]} \arrow[r, "l\circ -"] \arrow[d, "id_*"] \arrow[dr, phantom, "\text{② see rmk}"] & {[W, F(X/A, S^0)]} \arrow[r] \arrow[d, "f_*"] \arrow[dr, phantom, "\text{③ see rmk}"] & {[W, F(X,S^0)]} \arrow[r] \arrow[d, "id_*"] & {[W, F(A,S^0)]} \arrow[d, "id_*"] \\
{[W, \Sigma^{-1}F(X,S^0)]} \arrow[r] & {[W, \Sigma^{-1}F(A,S^0)]} \arrow[r, "k\circ -"] & {[W, \Sigma^{-1}F(A,S^0)/F(X,S^0)]} \arrow[r] & {[W, F(X,S^0)]} \arrow[r] & {[W, F(A,S^0)]}
\end{tikzcd}$$

The bottom line is exact since all three terms of

$$\Sigma^{-1}F(X,S^0) \to \Sigma^{-1}F(A,S^0) \to \Sigma^{-1}F(A,S^0)/F(X,S^0) \to F(X,S^0) \to F(A,S^0)$$ are cofiber seq.

Then by 5 Lemma, f_* is iso.

[Rmk] Why diagram ② and ③ are commutative?

It suffices to show diagram

$$\begin{tikzcd} \Sigma^{-1}F(A,S^0) \arrow[r, "\iota"] \arrow[dr, "k"'] & F(X/A,S^0) \arrow[d, "f"] \\ & \Sigma^{-1}F(A,S^0)/F(X,S^0) \end{tikzcd} \qquad \begin{tikzcd} F(X/A,S^0) \arrow[r, "h"] \arrow[d, "f"] & F(X,S^0) \\ \Sigma^{-1}F(A,S^0)/F(X,S^0) \arrow[ur, "g"'] & \end{tikzcd}$$

are commutes.

$$\begin{tikzcd} \Sigma^{-1}F(A,S^0) \arrow[r] \arrow[dr, "k"'] & F(X/A,S^0) \arrow[r, "h"] \arrow[d, "f"] & F(X,S^0) \arrow[r] & F(A,S^0) \\ & \Sigma^{-1}F(A,S^0)/F(X,S^0) \arrow[ur, "g"'] & & \end{tikzcd}$$

seq is a fiber seq.

So $\exists ! f, k$ s.t. diagram commutes.

[Fact] $A \to X \to Y := X \cup_f CA$ is a cofiber seq s.t. A and X are dualizable, then Y is dualizable and $DY \to DX \to DA$ is a fiber seq.

[Thm] Let X be a finite CW spectrum. The dual of X exists.

Pf: By $\eta: S^n \wedge S^{-n} \cong \Sigma^n S^{-n} \xrightarrow{\cong} S^0$, $\varepsilon: S^0 \xrightarrow{\cong} \Sigma^n S^{-n} \cong S^{-n} \wedge S^n$, we have $DS^n = S^{-n}$. S^n, D^{n+1} has dual, so $D^{n+1} \cup_f S^n$ has dual where $f: S^n \longrightarrow D^{n+1}$ by above [Fact]. Hence by induction, finite CW complexes have dual. Then $D\Sigma X = \Sigma^{-1} DX$. Then duals of finite spectra exist.

§ Alexander duality

Assume X be a finite CW complex embedded cellularly into S^n.

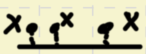
Let $A \subset S^n - X$ be a finite CW complex s.t. $A \sim S^n - X$.

[Exp] $X = S^{p-1}$ with $S^{p-1} \longrightarrow S^{p+q-1}$. Let $A = S^{q-1} \subseteq S^{p+q-1} - S^{p-1}$

[Def] Define $X * A := X \times [0,1] \times A / \sim$ where $\begin{cases} (x, 0, a_1) \sim (x, 0, a_2) \\ (x_1, 1, a) \sim (x_2, 1, a) \end{cases}$

called join of X and A.

[Rmk] At $t=0$  assign each $x \in X$ an A

At $t=1$  assign each $a \in A$ an X

[Fact] Join $X * A$ can be expressed by push out:

$$\begin{tikzcd} X \times A \arrow[r] \arrow[d] & CX \times A \arrow[d] \\ X \times CA \arrow[r] & X * A \end{tikzcd}$$

$CX := X \times I / X \times 1$ is the non-reduced cone

[prop] $X * A$ homotopy equiv to $\Sigma(X\wedge A)$.

pf: We have commutative diagram:

$$\begin{tikzcd} X \arrow[d, "id"'] & X\vee A \arrow[l] \arrow[r] \arrow[d] & A \arrow[d, "id"] \\ X \arrow[d] & X\times A \arrow[l] \arrow[r] \arrow[d] & A \arrow[d] \\ * & X\wedge A \arrow[l] \arrow[r] & * \end{tikzcd}$$

By take homotopy push out of all rows, we have cofiber seq $* \to ? \to \Sigma(X\wedge A)$

$$\begin{tikzcd} X\vee A \arrow[r] \arrow[d] & A \arrow[d] \\ X \arrow[r] & \Sigma(X\wedge A) \end{tikzcd} \qquad \begin{tikzcd} X\wedge A \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X\times A \arrow[r] \arrow[d] & A \arrow[d] \\ X \arrow[r] & ? \end{tikzcd}$$

With L.E.S. of fiber seq $* \to ? \to \Sigma(X\wedge A)$ on π_*, we have $? \to \Sigma(X\wedge A)$ induces weak homotopy equiv. X, A are CW complexes, so $? \to \Sigma(X\wedge A)$ is homotopy equiv. On the other hand,

the push out $\begin{tikzcd} X\times A \arrow[r] \arrow[d] & A \arrow[d] \\ X \arrow[r] & ? \end{tikzcd}$ is homotopy equiv to $\begin{tikzcd} X\times A \arrow[r] \arrow[d] & A\times CX \arrow[d] \\ X\times CA \arrow[r] & X * A \end{tikzcd}$

So $? \simeq X * A$. Therefore $X * A \to \Sigma(X\wedge A)$ is a homotopy equiv.

In the following X, Y, Z are spectrum.

[prop] Let Z be a spectrum and Y be a finite spectra. Then $DY\wedge Z \simeq F(Y,Z)$

pf: $[X, DY\wedge Z] = [X, Z\wedge DY] = [X\wedge Y, Z] = [X, F(Y,Z)] \Rightarrow DY\wedge Z \simeq F(Y,Z)$

[prop] $\pi_r F(Y,Z) = [Y,Z]_r$

pf: $\pi_r F(Y,Z) \underset{\pi_r E = [\mathbb{S}, E]_r}{=} [\mathbb{S}, F(Y,Z)]_r = [\Sigma^r \mathbb{S}, F(Y,Z)] = [\mathbb{S}, F(Y,Z)]_r$

$= [\mathbb{S}\wedge Y, Z]_r = [Y, Z]_r$

$\mathbb{S}\wedge Y = Y$ (sometimes write $\mathbb{S}$ as S^0)

[prop] Let Y be a finite spectrum. Suppose $[Y, H\mathbb{Z}]_r = 0$ for $\forall r\in\mathbb{Z}$. Then $Y \simeq *$.

pf: $C = \operatorname{colim}_n C_n$ is finite, hence $\exists\, n$ s.t. Y_n contains representatives for all stable cells. $\exists$ a finite subcomplex $K \subset Y_n$ containing all the chosen representatives. $\Sigma^{-n}K$ is spectrum $\Sigma^{-n}\Sigma^\infty K$ with $(\Sigma^{-n}K)_m = \Sigma^{m-n}K$. Since $K \subseteq Y_n$, we have $\Sigma^{m-n}K \subset \Sigma^{m-n}Y_n = Y_m$, so there is an inclusion of cofinal subspectrum $\Sigma^{-n}K \to Y$. $\Sigma^{-n}K \simeq Y$ in stable homotopy cat.

$\begin{cases} \tilde{H}^{n-r}(K,\mathbb{Z}) \cong [K, H\mathbb{Z}]_{r-n} = [\Sigma^n K, H\mathbb{Z}] \xrightarrow[\text{is an inclusion of } Y]{[Y,H\mathbb{Z}]_r=0 \text{ and } \Sigma^n K} 0 \overset{\text{Hurewicz}}{\Longrightarrow} K \simeq * \\ \Sigma K \subset Y_{n+1} \text{ so we can always assume } K \text{ simply connected} \end{cases}$

[prop] If X is a CW complex, then $\pi_r(X\wedge H\mathbb{Z}) \simeq \tilde{H}_r(X, \mathbb{Z})$

pf: For any spectrum E, we have $E_r X = \pi_r(X\wedge E)$

[Thm] (Alexander duality) Let X be a finite CW complex embedded into S^n s.t. $S^n - X$ is homotopy equiv to a finite CW complex. Then $DX \simeq \Sigma^{-(n-1)}(S^n - X)$ is an iso in the stable homotopy cat. Furthermore, for any spectrum E,
$E^r X \cong E_{n-r-1}(S^n - X)$ and $E_r X \cong E^{n-1-r}(S^n - X)$.

[Exp] $\tilde{H}_r(X,\mathbb{Z}) \to \tilde{H}^{n-1-r}(S^n - X, \mathbb{Z})$ is an iso. $\square$

Pf for Alexander duality, part I: Suppose we've proved $DX \simeq \Sigma^{-(n-1)}(S^n - X)$, i.e., a map $DX \to \Sigma^{-(n-1)}(S^n - X)$ (1) being an iso.

Apply $\pi_r(1) \wedge E$:

$$\begin{array}{ccc}
\pi_r(DX \wedge E) & \xrightarrow{\simeq} & \pi_r(\Sigma^{-(n-1)}(S^n - X) \wedge E) \\
\| \; {\scriptstyle DY \wedge Z \simeq F(Y,Z)} & & \cong \; {\scriptstyle \pi_r(X \wedge E) = E_r(X)} \\
\pi_r(F(X,E)) & & E_r(\Sigma^{-(n-1)}(S^n - X)) \\
\| \; {\scriptstyle \pi_r F(Y,Z) = [Y,Z]_r} & & \| \; {\scriptstyle E_n \Sigma \simeq E_{n-1}} \\
[X,E]_r = E^{-r}X & & E_{n-1+r}(S^n - X)
\end{array}$$

So it's $E^r X \cong E_{n-r-1}(S^n - X)$ (2).

Switch $S^n - X$ and X, r and $n-1-r$ in (2), we have $E^{n-1-r}(S^n - X) \cong E_r(X)$ $\square$

To show $DX \simeq \Sigma^{-(n-1)}(S^n - X)$, it suffices to show $\Sigma^{-(n-1)}A \to F(X, S^0)$ is an iso, since $DX = F(X, S^0)$, $S^n - X \simeq A$.

[Fact] $\tilde{H}_r(X;\mathbb{Z}) \to \tilde{H}^{-r+n-1}(A;\mathbb{Z})$ is iso. (Proof in Lecture 9, Prop 1.4)
Thus we complete the proof of Alexandar duality.

[prop] $\Sigma^{-(n-1)} A \to F(X, S^0)$ is an iso in stable homotopy cat.

Pf: To show this map is iso in stable homotopy cat, it suffices to show it induces iso on π_*. We have cofiber seq

$\Sigma^{-(n-1)}A \to F(X,S^0) \to Y := F(X,S^0) \cup C(\Sigma^{-(n-1)}A)$. By L.E.S. of π_*, it suffices to show $\pi_r Y = 0$ for all r. We've shown if $[Y, H\mathbb{Z}]_r = 0$ for all r, then $Y \simeq *$.
So to show $\pi_r Y = 0, \forall r$, it suffices to show $[Y, H\mathbb{Z}]_r = 0, \forall r$.

$\Sigma^{-(n-1)}A \to F(X, S^0) \to Y$ is a cofiber seq

$\Downarrow$ Apply D

$DY \to X \to D\Sigma^{-(n-1)}A$ is a cofiber seq

$\Downarrow$ $- \wedge H\mathbb{Z}$

$DY \wedge H\mathbb{Z} \to X \wedge H\mathbb{Z} \to D\Sigma^{-(n-1)}A \wedge H\mathbb{Z}$ is a cofiber seq

$\Downarrow$ L.E.S. on π_*

$$\cdots \to \pi_{r-1}(D\Sigma^{-(n-1)}A \wedge H\mathbb{Z}) \to \pi_r(DY \wedge H\mathbb{Z}) \to \pi_r(X \wedge H\mathbb{Z}) \to \pi_r(D\Sigma^{-(n-1)}A \wedge H\mathbb{Z}) \to \cdots$$

$\pi_r(DY \wedge H\mathbb{Z}) = \pi_r(F(Y, H\mathbb{Z})) = [Y, H\mathbb{Z}]_r \overset{\text{w.t.s.}}{=\!=} 0$

It suffices to show it's iso for all r.

$\pi_r(X\wedge H\mathbb{Z}) = \tilde{H}_r(X;\mathbb{Z})$ and $\pi_r(D\Sigma^{-(n-1)}A\wedge H\mathbb{Z}) = \pi_r F(\Sigma^{-(n-1)}A, H\mathbb{Z})$
$= [\Sigma^{-(n-1)}A, H\mathbb{Z}]_r = \tilde{H}^{-r}(\Sigma^{-(n-1)}A;\mathbb{Z}) = \tilde{H}^{-r+n-1}(A;\mathbb{Z})$
So it suffices to show $\tilde{H}_r(X;\mathbb{Z}) \to \tilde{H}^{-r+n-1}(A;\mathbb{Z})$ is iso, which is the [Fact].

§ Atiyah Duality

Recall: X is a finite CW complex. $DX \simeq \Sigma^{-(n-1)}(S^n - X)$, where $X\to S^n$ is any non-surj embedding of X, and this embedding always exists.

[Construction] Since $X\to S^n$ is non-surj, we can remove a point in S^n and add this point to X. Then we have $X' = X_+$ and $S^n - X' = S^n - X_+ = S^n - p - X = \mathbb{R}^n - X$
So the formula becomes $\underline{D(X_+)\simeq\Sigma^{-(n-1)}(\mathbb{R}^n - X)\simeq\Sigma^{-(n-1)}(D^n - X)}$ where D^n is the open n-disk in $\mathbb{R}^n$.
Actually, with cofiber seq we can compute $\Sigma^{-(n-1)}(\mathbb{R}^n - X)$. Consider cofiber seq

$$\mathbb{R}^n - X \longrightarrow \mathbb{R}^n \to \Sigma(\mathbb{R}^n - X)$$

(This cofiber seq is obtained by $\mathbb{R}^n - X \xrightarrow{id} \mathbb{R}^n - X \to \mathbb{R}^n - X/_{\mathbb{R}^n - X} = * \simeq \mathbb{R}^n$ in stable homotopy cat. So we have $\mathbb{R}^n - X \to \mathbb{R}^n - X \to \mathbb{R}^n \to \Sigma\mathbb{R}^n - X \to \cdots$ all three terms are cofiber seq. Hence $\mathbb{R}^n - X \to \mathbb{R}^n \to \Sigma(\mathbb{R}^n - X)$ is cofiber seq)

Extend it to left we obtain cofiber seq $\Sigma^{-n}(\mathbb{R}^n - X)\to\Sigma^{-n}\mathbb{R}^n \to \Sigma^{-(n-1)}(\mathbb{R}^n - X)$
(or apply Σ^{-n}) is also a fiber seq. Hence $\Sigma^{-(n-1)}(\mathbb{R}^n - X) = \Sigma^{-n}\mathbb{R}^n/\Sigma^{-n}(\mathbb{R}^n - X) = \Sigma^{-n}(\mathbb{R}^n/\mathbb{R}^n - X)$.
Let N be any n.b.h. of X in $\mathbb{R}^n$, we have $D(X_+)\simeq\Sigma^{-n}(\mathbb{R}^n/(\mathbb{R}^n - X))\simeq\Sigma^{-n}N/(N-X)$

[Thm] Let X be a compact mf without boundary, then $D(X_+)\simeq Th(-T_X)$.

Pf:

[Fact] If X is a mf, then N can be identify with $D(N_X\mathbb{R}^n)$ (disk bdl of normal bdl $N_X\mathbb{R}^n$) and ∂N identified with $S(N_X\mathbb{R}^n)$ (sphere bdl). Such n.b.h. N is called tubular n.b.h.

$$N/(N-X) = N/\partial N \overset{\text{[Fact]}}{=} D(N_X\mathbb{R}^n)/S(N_X\mathbb{R}^n) = Th(N_X\mathbb{R}^n)$$

$N_X\mathbb{R}^n \oplus T_X = \underline{\mathbb{R}}^n$

So $D(X_+) = \Sigma^{-n}(N/(N-X)) = \Sigma^{-n}Th(N_X\mathbb{R}^n) = Th(N_X\mathbb{R}^n - \underline{\mathbb{R}}^n) = Th(-T_X)$

($\Sigma^n Th(V) = Th(V\oplus\underline{\mathbb{R}}^n)$)

[Summary] We first rewrite Alexander duality as $D(X_+)\simeq\Sigma^{-n}(N/(N-X))$, represented by n.b.h. of X. Then use tubular n.b.h., we can rewrite $D(X_+)$ by <u>Thom space</u>. With Thom space, there are a lot of tools we can use, such as Thom iso thm.

[Fact] (Thom iso thm) If $V\to X$ is an orientable vect. bdl. of rank n, then there is a natural iso $\tilde{H}^{*+n}(Th(V);\mathbb{Z})\simeq H^*(X;\mathbb{Z})$.

[Rmk] One way to think about Thom iso thm is: V oriented $\Rightarrow$ V behaves like trivial bdl after taking cohomology, i.e., $\tilde{H}^{*+n}(Th(V);\mathbb{Z})\simeq\tilde{H}^{*+n}(Th(\underline{\mathbb{R}}^n);\mathbb{Z})$
$\simeq\tilde{H}^{*+n}(\Sigma^n X_+;\mathbb{Z})\simeq\tilde{H}^*(X_+;\mathbb{Z}) = H^*(X;\mathbb{Z})$

($\Sigma^n X_+ \simeq Th(\underline{\mathbb{R}}^n)$)

[Fact](Generalization of Thom iso thm) For any ab grp A, we have a nat. iso $\tilde{H}^{*+n}(Th(V);A) \cong H^*(X;A)$ when V is A-oriented.

[Fact](Generalization) For any spectrum E, we have a nat. iso
$E^{*+n}Th(V) \cong E^*(X_+)$ when V is E-oriented. (discuss later)

[Rmk] If V is E-orientable, cohomolog of $Th(V)$ and X_+ has relation.

[Thm] (Poincaré duality) Let X be a compact n-mf without boundary and E is a spectrum s.t. T_X is E-orientable. Then $E_*(X_+) \cong E^{n-*}(X_+)$.

pf:

$$E_*(X_+) = \pi_*(X_+ \wedge E) = [\Sigma^* S^0, X_+ \wedge E] = [S^0, X_+ \wedge E]_*$$

$$= [S^0 \wedge D(X_+), E]_r = [D(X_+), E]_r$$

Thom iso $E^{*+n}(ThV) = E^*(X_+)$

$$E^{n-*}(X_+) = E^{-*-(-n)}(X_+) = E^{-*-Th(-T_X)}(X_+) = E^{-*}Th(-T_X)$$

$D(X_+) \cong Th(-T_X)$

$$= E^{-*}D(X_+) = [D(X_+), E]_* = E_*(X_+)$$

□

[Exp] If X is orientable compact n-mf without bd, then

$$H_*(X;\mathbb{Z}) \cong \tilde{H}_*(X_+;\mathbb{Z}) \cong \tilde{H}^{n-*}(X_+;\mathbb{Z}) \cong H^{n-*}(X;\mathbb{Z}),$$ Poincaré Duality

[Notation] $\mathbb{D}(-)$ means duality and $D(-)$ means disk bdl.

[Thm] (Atiyah duality)

1. If X is a compact mf with boundary ∂X then $\mathbb{D}(X/\partial X) \cong Th(-T_X)$
2. If X is a compact mf without boundary and V is a sm vect. bdl., then $D(Th(V)) \cong Th(-T_X - V)$

(proof & Example truncated projective space is in Lecture 10)

§ Generalized homology

[Construction] (Natural abelian str on $[X,Y]_r$)

Given $f, g \in [X,Y]_r$, we have $f \vee g \in [X \vee X, Y]_r$. With the pinch map $p: X \to X \vee X$, we define $f+g = (f \vee g) \circ p \in [X,Y]_r$.

This makes $[X,Y]_r$ a grp. for example, $f+(-f)=0$ because

$$\Sigma\Sigma^{-1}X \xrightarrow{p} \Sigma\Sigma^{-1}X \vee \Sigma\Sigma^{-1}X \xrightarrow{f \vee (-f)} \Sigma\Sigma^{-1}X$$ is null.

$X \simeq \Sigma^n \Sigma^{-n} X$, so we can pinch along different suspension coordinate. With Eckmann-Hilton argument, these operations are the same.

[Fact] (Eckmann-Hilton) suppose X is a set equipped with two binary operations $+$ and $*$ s.t. each operation has an identity and for $\forall a,b,c,d \in X$ we have $(a+b)*(c+d) = (a*c)+(b*d)$. Then $+$ and $*$ are equal and both commutative and associate.

[Def] Let E be a spectrum. The generalized E-homology of degree r is the functor from stable homotopy cat to abelian grps

$$X \longmapsto E_*(X) := \pi_*(E \wedge X)$$

[Def] A generalized reduced homology theory on based CW-complexes is a sequence of functors $\tilde{h}_* : \mathrm{hoCW}_* \to \mathrm{Ab}$ from the homotopy cat of CW-complexes equipped with a base point to abelian grps for $* \in \mathbb{Z}$ s.t.

1. There are natural boundary maps $\tilde{h}_*(X/Y) \to \tilde{h}_{*-1}(Y)$ s.t.
$$\cdots \to \tilde{h}_*(Y) \to \tilde{h}_*(X) \to \tilde{h}_*(X/Y) \to \tilde{h}_{*-1}(A) \to \cdots$$
is exact for each CW pair (X,Y).

2. The map $\bigoplus_{a\in A} \tilde{h}_*(X_a) \to \tilde{h}_*(\bigvee_{a\in A} X_a)$ is an iso.

[Rmk] Generalized reduced homology theory $\tilde{h}_*$ on based CW-complexes can be extended to a functor on non-based CW-complexes by setting $\tilde{h}_*(X) = \ker(\tilde{h}_*(X_+) \to \tilde{h}_*(S^0))$. This extend homology on based CW complex is iso to the old. Indeed, a choice of CW complex $\Leftrightarrow$ a map $S^0 \to X_+$

one point of S^0 maps to disjoint base point of X_+ and the other choose a point in X

$\Leftrightarrow$ pick a base point in X $X_+ \cong X \vee S^0$ $\Leftrightarrow$ $\tilde{h}_*(X_+) \cong \tilde{h}_*(X \vee S^0) \cong \tilde{h}_*(X) \oplus \tilde{h}_*(S^0)$

This decomposition depends on $X \vee S^0$ so depends on map $S^0 \to X_+$

For X be a based CW complex, $\tilde{h}_*(X) = \ker(\tilde{h}_*(X_+) \to \tilde{h}_*(S^0))$
$= \ker(\tilde{h}_*(X) \oplus \tilde{h}_*(S^0) \to \tilde{h}_*(S^0)) = \underbrace{\tilde{h}_*(X)}_{\text{old } \tilde{h}_*}$

[prop] The functors $\mathrm{hoCW}_* \to \mathrm{Ab}$ defined by $X \mapsto E_*(\Sigma^\infty X)$ defines a generalized reduced homology theory for $\forall E$ in stable homotopy cat.

Pf: For CW pair, we have $X \cup CY \cong X/Y$ so $\Sigma^\infty Y \to \Sigma^\infty X \to \Sigma^\infty X/Y$ is a cofiber seq. Apply $E \wedge -$, we obtain cofiber seq

$$E \wedge \Sigma^\infty Y \to E \wedge \Sigma^\infty X \to E \wedge (\Sigma^\infty X/Y).$$

Consider L.E.S. of above fiber seq, we have

$$\cdots \to \pi_*(E\wedge\Sigma^\infty Y) \to \pi_*(E\wedge\Sigma^\infty X) \to \pi_*(E\wedge\Sigma^\infty X/Y) \to \pi_{*-1}(E\wedge\Sigma^\infty Y) \to \cdots$$

i.e., L.E.S. $\cdots \to \tilde{h}_*(Y) \to \tilde{h}_*(X) \to \tilde{h}_*(X/Y) \to \tilde{h}_{*-1}(Y) \to \cdots$ proved (1).

Since $E\wedge(\bigvee_{a\in A}X_a)\cong\bigvee_{a\in A}(E\wedge X_a)$, we have $E_*(\bigvee_{a\in A}X_a)\cong\pi_*(\bigvee_{a\in A}(E\wedge X_a))$ by applying π_*.

$\pi_*(\bigvee_{a\in A}(E\wedge X_a))=\operatorname{colim}_{n\to\infty}\pi_{n+*}(\bigvee_{a\in A}(E\wedge X_a))_n$. An element in $\pi_{n+*}(\bigvee_{a\in A}(E\wedge X_a))_n$ is a map $S^{n+*}\longrightarrow\bigvee_{a\in A}(E\wedge X_a)_n$. Since S^{n+*} is compact, the image of $S^{n+*}\longrightarrow\bigvee_{a\in A}(E\wedge X_a)_n$ lies in $\bigvee_{a\in A'}(E\wedge X_a)_n$ for some finite A'.

Similarly, $S^{n+*}\times I$ is also compact, and the image $S^{n+*}\times I\to\bigvee_{a\in A}(E\wedge X_a)_n$ lies in $\bigvee_{a\in A''}(E\wedge X_a)_n$ for some A'' finite. Therefore,

$\pi_{n+*}(\bigvee_{a\in A}(E\wedge X_a))_n\cong\operatorname{colim}_{A'\subset A}\pi_{n+*}\left(\bigvee_{a\in A'}(E\wedge X_a)_n\right)$, where the colimit is taken over finite set A' of A. Thus, we have

$$E_*(\bigvee_{a\in A}X_a)\cong\operatorname{colim}_{n\to\infty}\operatorname{colim}_{A'\subset A}\pi_{n+*}\left(\bigvee_{a\in A'}(E\wedge X_a)_n\right)=\operatorname{colim}_{A'\subset A}\operatorname{colim}_{n\to\infty}\pi_{n+*}\left(\bigvee_{a\in A'}(E\wedge X_a)_n\right)$$

$$=\operatorname{colim}_{A'\subset A}\pi_*(\bigvee_{a\in A'}(E\wedge X_a))=\operatorname{colim}_{A'\subset A}\bigoplus_{A'}\pi_*(E\wedge X_a)=\operatorname{colim}_{A'\subset A}\bigoplus_{A'}E_*(X_a)=\bigoplus_{A}E_*(X_a)$$

There is cofiber seq $E\wedge X\to(E\wedge X)\vee(E\wedge Y)\longrightarrow E\wedge Y\cong(E\wedge X)\vee(E\wedge Y)/(E\wedge X)$.
L.E.S. on π_* of this cofiber seq splits into short exact seq with $\pi_*(E\wedge Y)\longrightarrow\pi_*((E\wedge X)\vee(E\wedge Y))$ induced by $E\wedge Y\hookrightarrow(E\wedge X)\vee(E\wedge Y)$.
Hence $\pi_*((E\wedge X)\vee(E\wedge Y))\cong\pi_*(E\wedge X)\oplus\pi_*(E\wedge Y)$.

Actually, the inverse of above [prop] is also true.

[Thm] Let E be a spectrum and $\tilde{h}_*$ be a generalized reduced homology theory. $f:E_*\longrightarrow\tilde{h}_*$ is a map of homology theory. Then $\exists$ spectrum F with iso $F_*\cong\tilde{h}_*$ and a map $E\to F$ with induced map $E_*\to F_*\cong\tilde{h}_*$ is f.

[Rmk] This [Thm] shows any generalized reduced homology $\tilde{h}_*$ can be expressed by F_* for some spectrum F.

[Fact] Let A be an ab grp and HA be Eilenberg-Maclane spectrum. There is a nat iso between singular reduced homology with coefficients in A and the generalized homology of HA:

$$HA_*X\cong\tilde{H}_*(X;A)$$

On the left hand side, we may choose any 0 simplex of X as base pt to take the suspension spectrum.

§ Spectral sequence

[Def] A differential grp (E,d) is a grp together with a mor $d: E \to E$ with $d^2=0$. The homology $H(E,d)$ of (E,d) is $H(E,d) := \ker d / \operatorname{Im} d$.

[Rmk] Similarly we can define differential alg, differential module, differential vect. space.

[Def] A spectral seq is a seq of differential grps (E_n, d_n) for $n \geq 2$ (or 1 or 0) s.t. $E_n \cong H(E_{n-1}, d_{n-1})$.

[Rmk] It's "one-term-version" Serre spectral seq. Serre spectral seq $E_n^{p,q} \leftarrow \mathbb{Z}\oplus\mathbb{Z}$ terms
spectral seq $E_n \rightsquigarrow$ only one term.

[Rmk] Spectral seq is useful when E_2 can be computed and E_n's stabilize to some E_∞ that we wish to compute.

[Construction] (Spectral seq associated to an exact couple) An exact couple is a pair E, F of grps and a diagram
$$\begin{tikzcd} F \arrow[rr, "i"] & & F \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$
which is exact. Let $d = jk$, then

$d^2 = jkjk = j0k = 0$. So (E,d) is a differential grp. Now the goal is to construct an exact couple from the old. Let $E_1 = H(E,d)$. We want to find other terms in exact couple
$$\begin{tikzcd} F_1 \arrow[rr, "i_1"] & & F_1 \arrow[dl, "j_1"] \\ & E_1 \arrow[ul, "k_1"] & \end{tikzcd}$$
. $k_1 : \ker jk / \operatorname{Im} jk \to F_1$, so $k_1 jk = 0$ we find $kjk=0$,

so let k_1 be $k : \ker jk/\operatorname{Im} jk \to F_1$. $\operatorname{Im} k_1 = \operatorname{Im} k = \{kf \mid f \in \ker jk/\operatorname{Im} jk\} = \{kf \mid jkf = 0\}$
$= \ker j \cap \operatorname{Im} k = \operatorname{Im} i \cap \operatorname{Im} k$

so $\operatorname{Im} k_1 = \operatorname{Im} k \subseteq \operatorname{Im} i$. Since $F_1 \supseteq \operatorname{Im} k_1$, we can set $F_1 = \operatorname{Im} i$ and $i_1 = i|_{\operatorname{Im} i}$.

$\ker i_1 = F_1 \cap \ker i = \operatorname{Im} i \cap \ker i = \operatorname{Im} i \cap \operatorname{Im} k = \operatorname{Im} k = \operatorname{Im} k_1$. So it's exact at left F_1.

$$\begin{tikzcd} \operatorname{Im} i \arrow[rr, "i|_{\operatorname{Im} i}"] & & \operatorname{Im} i \arrow[dl, "j_1"] \\ & \ker jk/\operatorname{Im} jk \arrow[ul, "k"] & \end{tikzcd}$$

$\ker j_1 = \operatorname{Im} i_1 = \{ i^2 g \mid g \in F\} \rightsquigarrow j_1 i^2 = 0$.

Since $ji = 0$, we can set $j_1 = j i^{-1}$ where i^{-1} means preimage of map $i: F \to F$. Check j_1 is well-defined. $j_1 g = j i^{-1} g$. Suppose

$f_1, f_2 \in F$, $if_1 = if_2 = g \Rightarrow i(f_1 - f_2) = 0 \Rightarrow f_1 - f_2 \in \ker i = \operatorname{Im} k \Rightarrow jf_1 - jf_2 = j(f_1 - f_2) \in \operatorname{Im} jk$

$\ker j_1 = \{ g \in \operatorname{Im} i \mid j i^{-1} g = 0\} = \{ g \in \operatorname{Im} i \mid \exists f' \in F, if' = g, jf' = 0\} = \{g \in \operatorname{Im} i \mid \exists f' \in \underbrace{F \cap \ker j}_{\ker j = \operatorname{Im} i}, g = if'\}$

$= \{ g \in \operatorname{Im} i \mid \exists f' \in \operatorname{Im} i, g = if'\} = \{ g \in \operatorname{Im} i_1\} = \operatorname{Im} i_1$. So it's exact at right F_1.

Similarly $\ker k = \operatorname{Im} j_1$.

This diagram is exact, so it's an exact couple, also called the derived couple.

Repeating this process, we obtain a spectral seq (E_n, d_n).

What's the terms of spectral seq associated to exact couple explicitly?

$k_n = k$, $i_n = i|_{F_n}$, $j_n = j i^{-n}$, $F_n = \operatorname{Im} i^n$, $E_n = H(E_{n-1}, d_{n-1})$, $d_n = j_n k$. $\square$

Next, we come to other construction of spectral seq. We first study what's convergence. Naively, (E_n, d_n) converges to grp A if $E_n = A$ for sufficiently large n. But it's not sufficiently general to be useful.

Let A be a filtered grp with filtration $\cdots \subset \mathcal{F}_n A \subset \mathcal{F}_{n+1} A \subset \cdots$ where $n \in \mathbb{Z}$. Assume $\bigcup \mathcal{F}_n A = A$ and $\bigcap \mathcal{F}_n A = 0$.

[Def] The associated graded of a filtered grp A is $\quad A = \bigoplus_p \mathcal{F}_{p+1} A / \mathcal{F}_p A$

(Frequently holds) $\quad A = \bigoplus_n \mathcal{F}_{n+1} A / \mathcal{F}_n A$

[Def] If (E_n, d_n) are graded, we say (E_n, d_n) converges to A, $(E_n, d_n) \Rightarrow A$, if for each homogenous piece $(E_n)_p$, it stabilize to the associated graded of A, i.e., $(E_\infty)_p = \mathcal{F}_{p+1} A / \mathcal{F}_p A$.

[Rmk] $(E_n, d_n) \Rightarrow A$ is not only grp $E_\infty = A$, but only converge in graded level — the graded must also converges to graded of A, which is $A = \bigoplus_p \mathcal{F}_{p+1} A / \mathcal{F}_p A$.

[Rmk] We only define graded case, the bigraded case is the same.

Next, we'll construct spectral seq from double complex.

[Def] A double complex is a bunch of grps $A^{p,q}$ for $p, q \in \mathbb{Z}_{\geq 0}$ and differentials $d: A^{p,q} \to A^{p-1,q}$, $d': A^{p,q} \to A^{p,q-1}$ s.t. $dd' = d'd$ or $dd' = -d'd$.
(Choose one)

[Rmk] For two cases commute & anti-commute, we can go from one to the other by changing d' to $(-1)^p d'$. $\square$

Assume $dd' = -d'd$.

[Def] Define $\operatorname{Tot} A^{*,*}$ to be the differential graded grp $\operatorname{Tot} A^{*,*} = \bigoplus_n \left(\bigoplus_{p+q=n} A^{p,q} \right)$ with differential $D = d + d'$.

[prop] There are two natural spectral seq's associated to double complex and both converge to $H(\operatorname{Tot} A^{*,*}, D)$.

The following we'll show this.

[Construction] (Spectral seq associated to double complex)

Define a filtration $\mathcal{F}_n := \mathcal{F}_n \operatorname{Tot} A^{*,*} = \bigoplus_{p \leq n} A^{p,q}$ s.t. $D: \mathcal{F}_n \to \mathcal{F}_n$.

There are short exact seq's for each n:

$$0 \to \mathcal{F}_n \to \mathcal{F}_{n+1} \to \mathcal{F}_{n+1}/\mathcal{F}_n \to 0$$

$$\Big\Downarrow \text{ L.E.S. on } H_*$$

$$\cdots \to H_i(\mathcal{F}_n) \xrightarrow{f_i^n} H_i(\mathcal{F}_{n+1}) \xrightarrow{g_i^n} H_i(\mathcal{F}_{n+1}/\mathcal{F}_n) \xrightarrow{\Delta_i} H_{i-1}(\mathcal{F}_n) \to \cdots \qquad (1)$$

The idea is construct an exact couple b (1) and then obtain a spectral seq.

Equivalently, (1) can be rewritten as
$$\begin{tikzcd} H_*(\mathcal{F}_n) \arrow[rr] & & H_*(\mathcal{F}_{n+1}) \arrow[dl] \\ & H_*(\mathcal{F}_{n+1}/\mathcal{F}_n) \arrow[ul, "\partial"] & \end{tikzcd}$$
, which looks as exact couple!

This motivates us to define $F_1 = \bigoplus_{p,q} H_{p+q}\mathcal{F}_p$, $E_1 = \bigoplus_{p,q} H_{p+q}(\mathcal{F}_p/\mathcal{F}_{p-1})$.
Define i_1, j_1, k_1 using maps from L.E.S. :

① define $i_1 : \bigoplus_{p,q} H_{p+q}\mathcal{F}_p \to \bigoplus_{p,q} H_{p+q}\mathcal{F}_p$ by data $H_{p+q}\mathcal{F}_p \xrightarrow{f^p_{p+q}} H_{p+q}\mathcal{F}_{p+1}$;

② define $j_1 : \bigoplus_{p,q} H_{p+q}\mathcal{F}_p \longrightarrow \bigoplus_{p,q} H_{p+q}(\mathcal{F}_p/\mathcal{F}_{p-1})$ by data $H_{p+q}(\mathcal{F}_p) \xrightarrow{g^{p-1}_{p+q}} H_{p+q}(\mathcal{F}_p/\mathcal{F}_{p-1})$

③ define $k_1 : \bigoplus_{p,q} H_{p+q}(\mathcal{F}_p/\mathcal{F}_{p-1}) \longrightarrow \bigoplus_{p,q} H_{p+q}(\mathcal{F}_p)$ by data $H_{p+q}(\mathcal{F}_p/\mathcal{F}_{p-1}) \xrightarrow{\Delta^{p-1}_{p+q}} H_{p+q-1}(\mathcal{F}_{p-1})$

Since (1) is exact,
$$\begin{tikzcd} F_1 \arrow[rr, "i_1"] & & F_1 \arrow[dl, "j_1"] \\ & E_1 \arrow[ul, "k_1"] & \end{tikzcd}$$
is an exact couple.

There is another spectral seq obtained by switching the roles of p and q in the definition of filtration, i.e., $\mathcal{F}_n := F_n \operatorname{Tot} A^{*,*} = \bigoplus_{p\le n} A^{p,q}$ & $\mathcal{F}_n := \bigoplus_{q\le n} A^{p,q}$ leads to two spectral seq.

page 4 in Lecture 12. show this two spectral seq converges to the same grp.

§Complex K-theory

Notation: X is a finite CW-complex. Let $\operatorname{Vect}(X)$ be the set of iso classes of complex vector bdls on X. $\oplus$ is the Whitney sum, an operation on $\operatorname{Vect}(X)$.

[Def] $K^0(X) = \{(V,W) \in \operatorname{Vect}(X) \times \operatorname{Vect}(X)\}/\sim$ where $(V,W) \sim (V',W')$ iff $V \oplus W' \oplus U \cong V' \oplus W \oplus U$ for some $U \in \operatorname{Vect}(X)$. The operation $\oplus$ is $(V,W) \oplus (V',W') = (V\oplus V', W \oplus W')$.

[Rmk] 1. We can think (V,W) as "$V-W$".

2. $K^0(X)$ can also be defined by universal prop. $(K^0(X), f: \operatorname{Vect}(X) \to K^0(X))$ is a pair, where f sends $\oplus$ on $\operatorname{Vect}(X)$ to grp operation on $K^0(X)$ s.t. for any $g: \operatorname{Vect}(X) \to K$ sending $\oplus$ to grp operation on K, $\exists!\, h$ s.t.
$$\begin{tikzcd} \operatorname{Vect}(X) \arrow[r, "f"] \arrow[dr, "g"'] & K^0(X) \arrow[d, "\exists h"] \\ & K \end{tikzcd}$$
diagram commutes.

image of V, W in $K^0(X)$

[Prop] For $V, W \in \operatorname{Vect}(X)$, $\tilde{V} = \tilde{W}$ in $K^0(X) \Leftrightarrow \exists n$ s.t. $V \oplus \underline{\mathbb{C}}^n \cong W \oplus \underline{\mathbb{C}}^n$ where $\underline{\mathbb{C}}$ denotes trivial bdl on X.

pf: Since $V=W$ in $K^0(X)$, $\exists U \in \operatorname{Vect}(X)$ s.t. $V\oplus U \cong W \oplus U$. X is compact, so $\exists$ a finite cover $\{\mathcal{U}_i\}$ of X s.t. over each $\mathcal{U}_i$, $U \to X$ is a trivial bdl $U_i \times \mathbb{C}^n \xrightarrow{\pi_i} U_i$, i.e., $\mathcal{U}_i \times \mathbb{C}^n \xrightarrow{p_i} \underline{\mathbb{C}}^n$ injectively. $\{\mathcal{U}_i \times \mathbb{C}^n\}$ is a covering

of U, so $\exists$ partition of unity $\varphi_i : U \to \underline{\mathbb{C}}^n$ with $\Sigma\varphi_i = id$, $\mathrm{supp}\varphi_i \subseteq U_i \times \mathbb{C}^n$.
Then $f := \Sigma \rho_i \cdot \varphi_i : U \to \underline{\mathbb{C}}^n$ is a map injectively. Choose inner product on $\underline{\mathbb{C}}^n$, the injection $U \to \underline{\mathbb{C}}^n$ is split, i.e., $U \oplus \ker f \cong \underline{\mathbb{C}}^n$.

$V \oplus U \cong W \oplus U \xrightarrow{-\oplus \ker f} V \oplus U \oplus \ker f \cong W \oplus U \oplus \ker f$ i.e., $V \oplus \underline{\mathbb{C}}^n \cong W \oplus \underline{\mathbb{C}}^n$

[Rmk] $(V, W) \cong (V', W') \Rightarrow \exists U$ s.t. $V \oplus W' \oplus U \cong V' \oplus W \oplus U$. $\exists U'$ s.t. $U \oplus U' \cong \mathbb{C}^N$ so $V \oplus W' \oplus \mathbb{C}^N \cong V' \oplus W \oplus \mathbb{C}^N$.

[prop] Every element of $K^0(X)$ can be represented as $(V, \underline{\mathbb{C}}^n)$.
pf: For any $(V, W) \in K^0(X)$, by the same arguement above, $\exists W'$ s.t. $W \oplus W' \cong \underline{\mathbb{C}}^N$ for some N. Since $V \oplus \underline{\mathbb{C}}^N \cong V \oplus W' \oplus W$, we have $(V, W) \cong (V \oplus W', \underline{\mathbb{C}}^N)$.

[Coro] For $V, W \in \mathrm{Vect}(X)$ determine same class in $K^0(X)$, there is $\Sigma^\infty Th(V) \cong \Sigma^\infty Th(W)$ in the stable homotopy theory.
pf: $V = W$ in $K^0(X)$, then $\exists n$ s.t. $V \oplus \underline{\mathbb{C}}^n \cong W \oplus \underline{\mathbb{C}}^n$ by [prop].
Apply $Th(-)$, we have $Th(V \oplus \underline{\mathbb{C}}^n) \cong Th(W \oplus \underline{\mathbb{C}}^n)$, i.e., $\Sigma^{2n} Th(V) \cong \Sigma^{2n} Th(W)$.
Apply $\Sigma(-)$, we have $\Sigma\Sigma^{2n} Th(V) \cong \Sigma\Sigma^{2n} Th(W)$, i.e., $\Sigma^{2n+1} Th(V) \cong \Sigma^{2n+1} Th(W)$.
$\Sigma^\infty Th(V)$ and $\Sigma^\infty Th(W)$ have same spaces. The str map of "$\Sigma^\infty X$" are identity, so $\Sigma^\infty Th(V)$ & $\Sigma^\infty Th(W)$ have same str map. So $\Sigma^\infty Th V = \Sigma^\infty Th W$.

[Construction] (Thom spectrum $Th(V, W)$) Let $(V, W) \in K^0(X)$. Choose $(V', \underline{\mathbb{C}}^n)$ s.t. $(V, W) = (V', \underline{\mathbb{C}}^n)$. Define $Th(V, W) := \Sigma^{-2n} Th(V')$. It's well-defined. Indeed, if $(V_1, \underline{\mathbb{C}}^{n_1}) = (V_2, \underline{\mathbb{C}}^{n_2})$ in $K^0(X)$, then $V_1 \oplus \underline{\mathbb{C}}^{n_2} \cong V_2 \oplus \underline{\mathbb{C}}^{n_1}$. Apply $Th(-)$, $\Sigma^{2n_2} Th(V_1) = \Sigma^{2n_1} Th(V_2)$, i.e., $\Sigma^{-2n_1} Th(V_1) \cong \Sigma^{-2n_2} Th(V_2)$.

[Construction] (Contravariant $K^0(-)$) $f: X \to Y$ is a map of spaces, then we define $K^0(Y) \to K^0(X)$ where f^*V is the pullback of Y along f.

$(V, \underline{\mathbb{C}}^n) \mapsto f^*V$

[Construction] (reduced K^0, $\tilde{K}^0$) The reduced K^0 of X, $\tilde{K}^0(X)$, is defined as <u>either</u> a subgrp of $K^0(X)$ consists of (V, W) that V, W have the same rank.
<u>or</u> $\tilde{K}^0(X) = \mathrm{Vect}(X)/\sim$ where $V \sim W$ iff $\exists n, m$ s.t. $V \oplus \underline{\mathbb{C}}^n = W \oplus \underline{\mathbb{C}}^m$.

[Construction] (K-theory) For $n > 0$, $K^{-n}(X) = \tilde{K}^0(S^n \wedge X_+)$. For $n < 0$, we can use Bott Periodicity $K^n(X) \cong K^{n+2}(X)$.

§ KO theory
Notation: $\mathrm{Vect}(X)$ be the real vector bdls over X.

[Def] $KO^0(X)$ is the initial grp receiving a map from $\mathrm{Vect}(X)$ which sends $\oplus$ to grp operation on $KO^0(X)$.

[Prop] For $(V,W) \in KO^0(X)$, $\exists V', \mathbb{R}^n$ s.t. $(V,W) \cong (V', \mathbb{R}^n)$.

[Construction] (Thom spectrum $Th(V,W)$). Write $(V,W) = (V', \mathbb{R}^n)$. Define $Th(V,W) = \Sigma^{-n} Th(V')$ be an object in stable homotopy cat, where $Th(V')$ is spectrum $\Sigma^\infty Th(V')$. $\square$

Similary, let $KO^{-n}(X) = \widetilde{KO}^0(S^n \wedge X_+)$ for $n > 0$. It extends to minus n by real version of Bott periodicity $\widetilde{KO}^n(X) \cong \widetilde{KO}^{n+8}(X)$.