Homotopy Theory via Model Categories and their Underlying ∞-Categories

Zhou Fang

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Shenzhen Institute for Quantum Science and Engineering,
Southern University of Science and Technology

A categorical setting for doing topology

Ref: first lecture in `https://yuzhangmath.github.io/`, by Yu Zhang.

There are some choices for topologists doing topology.

- Work with **Top** whose objects are topology spaces and morphisms are continuous maps. (Not a good choice)
- Work with CW complexes (Not a good choice)
- Work with compactly generated weakly Hausdorff spaces (Nice)
- Work with simplicial sets (Nice)

The last choice is strange since a simplicial set is not a space!

You may understand why after this lecture.

Model categories

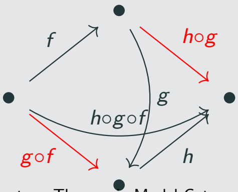
Usually, there is a class of morphisms that is unable to inverse but similar to isomorphism to some extent, e.g., weak homotopy equivalence between topological spaces.

There are some axioms to define such a class in a category $\mathcal{C}$. In **different occasion**, we **choose different axioms**.

Definition

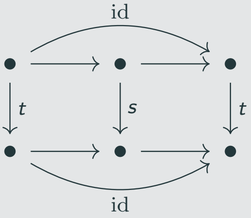
(Axioms for weak equivalence) Here are some commonly applied hypotheses for weak equivalence, denoted by $\mathcal{W}$.

- **Two-of-three**: Let $g : X \to Y$, $f : Y \to Z$ be two morphisms. If two of g, f, $f \circ g$ lie in $\mathcal{W}$, then so is the other.
- **Two-of-six**: For any composable triple of morphisms as follows, if $g \circ f, h \circ g \in \mathcal{W}$, then $f, g, h, h \circ g \circ f \in \mathcal{W}$.



Definition

- **Closed under retracts in the arrow category**: If $s \in \mathcal{W}$, then so is its retract t.



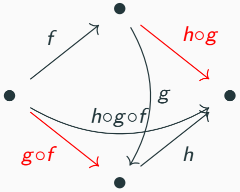
- It's reasonable to suppose $\mathcal{W}$ contains all isomorphisms. If not, at the bare minimum, $\mathcal{W}$ contains all identities.

A *homotopical category* is a pair $(\mathcal{C}, \mathcal{W})$ where $\mathcal{C}$ is a category and $\mathcal{W}$ is the weak equivalence (satisfying some of the above properties).

Replace all "$\in \mathcal{W}$" to isomorphism; these properties obviously hold. This shows **weak equivalence has the same behavior as isomorphism in this context**, although weak equivalence may not be an isomorphism.

Two-of-three: Let $g : X \to Y$, $f : Y \to Z$ be two morphisms. If two of g, f, $f \circ g$ are isomorphisms, then so is the other.

Two-of-six: For any composable triple of morphisms as follows, if $g \circ f$, $h \circ g$ are isomorphisms, then $f, g, h, h \circ g \circ f \in \mathcal{W}$.



The following provides a method to construct a class of morphisms automatically satisfying the above hypothesis of weak equivalences.

Property

Given a functor $F : \mathcal{C} \to \mathcal{D}$, let $\mathcal{W}$ **consists of morphims that inverted by functor** F, i.e.,

$$\mathcal{W} = \{f \in \mathrm{Mor}(\mathcal{C}) | F(f) \text{ is an isomorphims in } \mathcal{D}\}$$

Then $\mathcal{W}$ satisfies all of above hypothesis.

The main goal in this part is that the weak equivalence of the model category precisely consists of those morphisms inverted by the Gabriel-Zisman localization functor.

Definition

A *weak factorization system* $(\mathcal{L}, \mathcal{R})$ on a category $\boldsymbol{M}$ is comprised of two classes of morphisms $\mathcal{L}$ and $\mathcal{R}$ satisfying the following properties:

- **factorization**: For any $f \in \mathrm{Mor}(\boldsymbol{M})$, there exists $l \in \mathcal{L}$, $r \in \mathcal{R}$ such that $f = rl$
- **lifting property** Morphisms in $\mathcal{L}$ has left lifting property with respect to each morphims in $\mathcal{R}$.
- **closed under retratcts** $\mathcal{L}$ and $\mathcal{R}$ are each closed under retracts in the arrow category.

Property

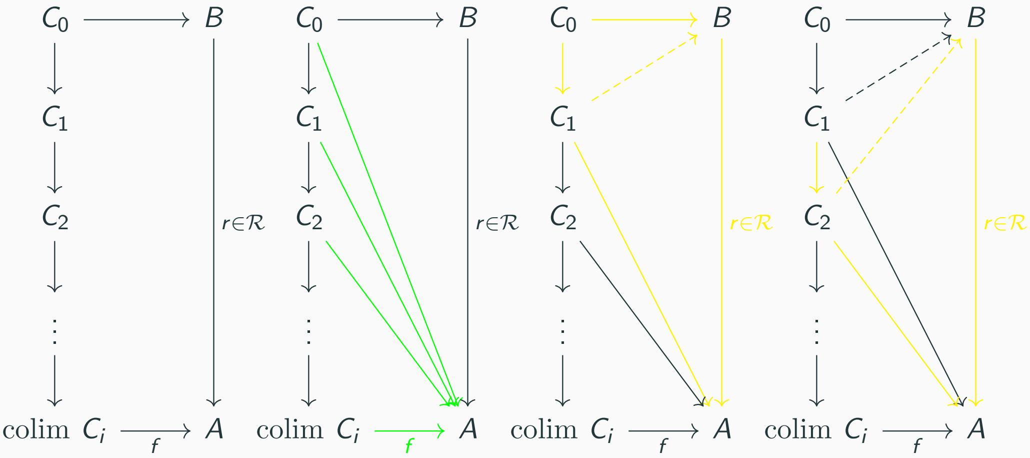
$\mathcal{L}$ contains the isomorphisms and is closed under coproduct, pushout, retract, and (transfinite) composition. $\mathcal{R}$ has similar closed properties.

All of the proof has the mode:

- Pick a clever diagram.
- Use universal property or lifting property to construct morphisms.

We only show $\mathcal{L}$ is closed under transfinite composition here; the others are similar.

The transfinite composition of an infinite composable morphisms $C_0 \xrightarrow{l_0} C_1 \xrightarrow{l_1} C_2 \to \cdots$ is the data $l : C_0 \to colimC_i$


C_0
B
C_1
C_2
$r \in \mathcal{R}$
colim C_i
f
A

For any following diagram with $[r : B \to A] \in \mathcal{R}$, we want to construct a lift $colimc_i \to B$.

$f : colimC_i \to A$ is equivalent to $C_i \to A$ such that diagram commutes.

Then we lift at each square:

When construct all $C_i \to B$, these morphisms are equivalent to $colimC_i \to B$, which is the lifting.

Definition

A model structure on homotopical category $(\boldsymbol{M}, \mathcal{W})$ consists of:

- **weak equivalence** $\mathcal{W}$, data from homotopical category. Morphisms in $\mathcal{W}$ are denoted by $\xrightarrow{\sim}$
- **cofibrations** $\mathcal{C}$, morphisms in $\mathcal{C}$ are denoted by $\rightarrowtail$
- **fibrations** $\mathcal{F}$, morphisms in $\mathcal{F}$ are denoted by $\twoheadrightarrow$

such that $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ define **weak factorzation systems** on $\boldsymbol{M}$. We refer to morphisms in $\mathcal{C} \cap \mathcal{W}$ as **trivial cofibrations**, denoted by $\overset{\sim}{\rightarrowtail}$ and morphisms in $\mathcal{F} \cap \mathcal{W}$ as **trivial fibrations**, denoted by $\overset{\sim}{\twoheadrightarrow}$.

Property

The left class $\mathcal{L}$ is determined by right class $\mathcal{R}$ in a factorization system: $\mathcal{L}$ consists of morphisms having left lifting property with respect to any morphisms in $\mathcal{R}$.

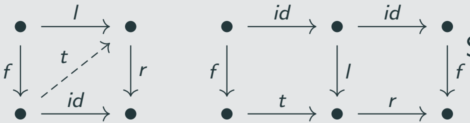
The right class $\mathcal{R}$ is determined by left class $\mathcal{L}$ in a factorization system: $\mathcal{R}$ consist of morphisms having right lifting property with respect to any morphisms in $\mathcal{L}$.

Remark

Model structure on **Top** is in Example 1.1.7 in [**Maciel**]**Maciel**.

Proof.

The left class consists of morphisms that have left lifting property with respect to the right class. Any left class has left lifting property with respect to the right class. It suffices to show that any f has left lifting property with respect to the right class lies in the left class. In particular, f has left lifting to its right factoring



So f is a retraction of l. Since the left class is closed under retract, so $f \in \mathcal{L}$. □

Corollary

A model structure on $\boldsymbol{M}$ (if exits) is uniquely determined by any two of the $\mathcal{C}$, $\mathcal{F}$, $\mathcal{W}$.

Gabriel-Zisman localization functor

Definition

Let **X** be the category consisting of the following data:

- $\mathrm{ob}(\mathbf{X}) = X_0, X_1$. We call X_1 the *object of vertices* and X_1 the *object of edges*.
- $\mathrm{Hom}_{\mathbf{X}}(X_0, X_0) = id$, $\mathrm{Hom}_{\mathbf{X}}(X_1, X_1) = id$, $\mathrm{Hom}_{\mathbf{X}}(X_0, X_1) = \emptyset$, $\mathrm{Hom}_{\mathbf{X}}(X_1, X_0) = s, t$. We call s the **source** and call t the **target**.

A **directed graph** is a functor $X \to \mathbf{Set}$.

Remark

- A direct graph can be viewed as a set of arrows from a given category.
- Directed graph in category theory $\neq$ directed graph in graph theory
- Let $\mathcal{C}, \mathcal{D}$ be two categories. A graph of shape $\mathcal{D}$ in $\mathcal{C}$ is a functor $\mathcal{D} \to \mathcal{C}$. When $\mathcal{D}$ is weird, it is not just "a set of arrows". So, the directed graph in category theory is much like a "graph" in graph theory.

A free category is freely generated by the composition of arrows.

Definition

A *free category* (also called *path category*) on a directed graph $G : X \to \mathbf{Set}$,denoted by PG is a category consists of following data:

- $\mathrm{ob}(PG) = X_0$
- For $x, y \in \mathrm{ob}(PG)$, let $\mathrm{Hom}_{PG}(x, y)$ consists of composable tuples in X_1, denoted as $(x; f_1, f_2, \cdots, f_n; x)$

Idea of constructing $\mathcal{C}[\mathcal{W}^{-1}]$

Tool	Usage
Directed graph and opposite category	Add inverse of morphism in $\mathcal{W}$ formally
free category	Make a directed graph a category

We all know $\mathcal{C}[\mathcal{W}^{-1}]$ is a category formally adding the inverse of morphisms in $\mathcal{W}$.
The key to achieving this is using the opposite category!

Construction

Let $\mathcal{C}$ be a category and $\mathcal{W}$ a class of morphism in $\mathcal{C}$. The category of fractions $\mathcal{C}[\mathcal{W}^{-1}]$ is obtained by the following steps:

- Let $\mathcal{W}^{op}$ denote the corresponding class of morphisms in $\mathcal{C}^{op}$.
- Let $G : X \to \mathbf{Set}$ where $G(X_0) = \text{ob}(\mathcal{C})$, $\boldsymbol{G(X_1) = \mathrm{Mor}(\mathcal{C}) \coprod_{x,y \in \mathrm{ob}(\mathcal{C})} \mathcal{W}^{op}(x,y)}$, where $\mathcal{W}^{op}(x,y)$ consists of all morphisms in $\mathcal{W}^{op}$ with x and y be its source and target, respectively. Hence, the morphism of G between x and y is $\mathcal{C}(x,y) \coprod \mathcal{W}^{op}(x,y)$. The arrows in $\mathcal{W}^{op}$ is denoted by $\bar{f}$ corresponding to $f \in \mathcal{W}$. The source and target map $G(s)$, $G(t)$ is obvious.
- We obtain a free category PG on G and we quotient it by the following relationship:
  - For any $x \in \text{ob}(\mathcal{C})$, $(x; id_x; x) \sim (x; \emptyset; x)$
  - For all $f : x \to y$, $g : y \to z$ in $\text{Mor}(\mathcal{C})$ $(x; f, g; z) \sim (x; gf; z)$
  - For all $f : x \to y$ in $\mathcal{W}$, $(x; f, \bar{f}; x) \sim (x; id_x; x)$ and $(y; \bar{f}, f; y) \sim (y; id_y; y)$

- There is an evident embedding: $\iota : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$
- There is a similar universal property analogous to fraction rings.

Property

Let $\mathcal{C}[\mathcal{W}^{-1}]$ be the category of fractions of category $\mathcal{C}$ and class of morphisms $\mathcal{W}$. For any category $\mathcal{D}$, $- \circ \iota : \mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is fully faithful embedding with essential image $\underset{\mathcal{W} \mapsto \cong}{\mathrm{Fun}}(\mathcal{C}, \mathcal{D})$, where $\underset{\mathcal{W} \mapsto \cong}{\mathrm{Fun}}(\mathcal{C}, \mathcal{D})$ consists of functors inverting $\mathcal{W}$. Hence, $\mathbf{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{D}) \cong \underset{\mathcal{W} \mapsto \cong}{\mathbf{Fun}}(\mathcal{C}, \mathcal{D})$ is isomorphism of categories.

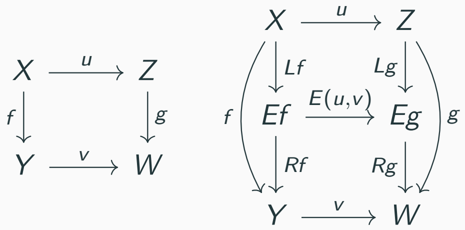
We let $\boldsymbol{M}$ be a model category and admit terminal object $*$ and initial object $\emptyset$. Idea: Use unique morphism $X \to *$ and $\emptyset \to X$ to define cofibrant/fibrant.

Definition

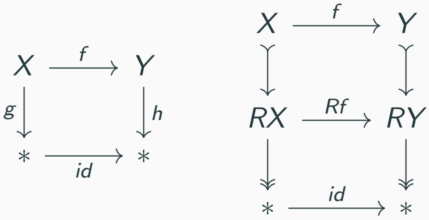
Let X be an object in a model category $\boldsymbol{M}$. Say X is:

- **fibrant** if $X \to *$ is a fibration;
- **cofibrant** if $\emptyset \to X$ is a cofibration.

Roughly speaking, a functorial factorization is a triple of functors $L, R : \boldsymbol{M}^2 \to \boldsymbol{M}^2$, $E : \boldsymbol{M}^2 \to \boldsymbol{M}$ such that for any commutative diagram on the left, there is a commutative diagram on the right.

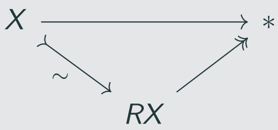


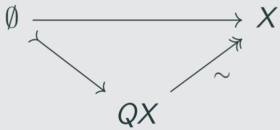
We always assume our model categories have functorial factorization by $(\mathcal{L}, \mathcal{R})$(Not true for all model categories, but it is difficult to find model categories that fail to satisfy this condition)



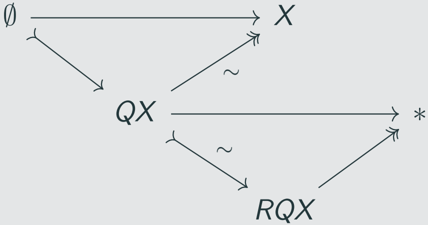
(We apply functorial factorization. Rg is uniquely determined by X, so denoted as RX; $R(f, id)$ is uniquely determined by f so denoted as Rf.) Hence, R **is a functor** by functorial factorization.

Construction





Obtain fibrant-cofibrant objects RQX or QRX, e.g.,



There exists a comparison weak equivalence:

$$RQX \xrightarrow{\sim} QRX$$

(The construction see page 13 in [**Riehl19**])

Construction

Fibrant and cofibrant replacement. Let $\boldsymbol{M}$ be a model category.

By functorial factorizations, we have:

- A **fibrant replacement functor** $R : \boldsymbol{M} \to \boldsymbol{M}$
- A **cofibrant replacement functor** $Q : \boldsymbol{M} \to \boldsymbol{M}$

equipped with **natural weak equivalences**(natural transformation and weak equivalence for each component):

$$\eta : \mathrm{id}_{\boldsymbol{M}} \overset{\sim}{\Longrightarrow} R \quad \text{and} \quad \epsilon : Q \overset{\sim}{\Longrightarrow} \mathrm{id}_{\boldsymbol{M}}$$

Homotopy theory by model categories

There are two choices for defining homotopy, called the "handness" of homotopy. This comes from **"dual"** in category theory.($A \coprod A$ is dual to $A \times A$)

Construction

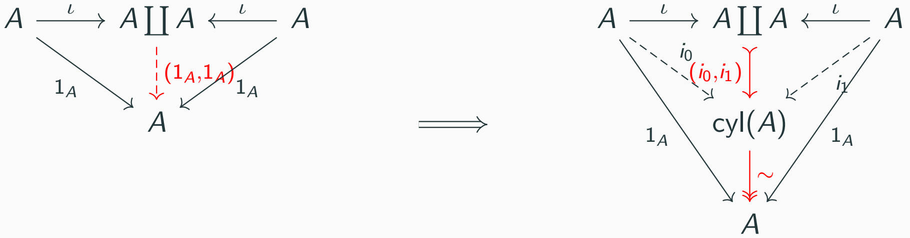
Let A be an object in a model category.

A cylinder object for A is given by a factorization of fold map

$$A \coprod A \xrightarrow{(1_A, 1_A)} A$$

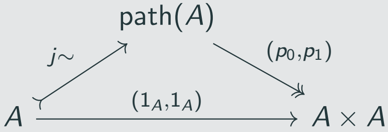
$$A \coprod A \overset{(i_0, i_1)}{\rightarrowtail} \mathrm{cyl}(A) \overset{q\sim}{\twoheadrightarrow} A$$

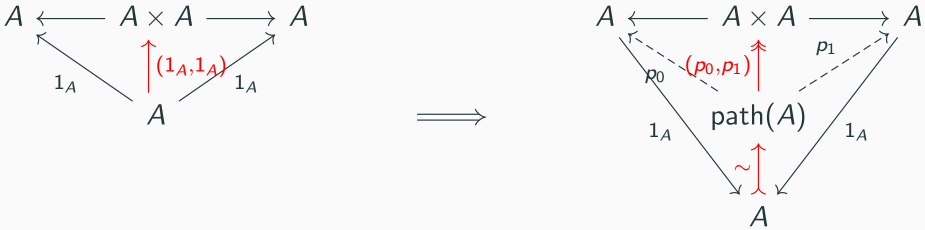
More explicitly,



Construction

A *path object* for A is given by a factorization of diagonal map





Definition

Let $f, g : A \to B$ be two morphisms in a model category $\boldsymbol{M}$. A **left homotopy** H from f to g is a morphism $H : \mathrm{cyl}(A) \to B$ such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{i_0} & \mathrm{cyl}(A) & \xleftarrow{i_1} & A \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle H} & \swarrow{\scriptstyle g} & \\ & & B & & \end{array}$$

If the left homotopy exists, we write $f \sim_l g$.

- $A \xrightarrow{i_0} \mathrm{cyl}(A) \xleftarrow{i_1} A$ is the data of cylinder object of A.
- The form $Hi_0 = Hi_1$ is similar to the ordinary homotopy theory

We can use left homotopy to define homotopy on spectra.

Define spectrum $Cyl(E)$ by $Cyl(E)_n := [0,1]_+ \wedge E_n$ with evident structure maps.

Define $i_0 : E \to Cyl(E)$ by $(i_0)_n : E_n \simeq \{0\}_+ \wedge E_n \hookrightarrow [0,1]_+ \wedge E_n$.

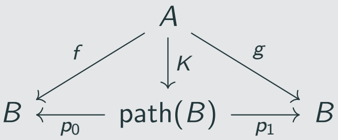
Define $i_1 : E \to Cyl(E)$ by $(i_1)_n : E_n \simeq \{1\}_+ \wedge E_n \hookrightarrow [0,1]_+ \wedge E_n$.

Two pmaps $f, g : E \to F$ are homotopic if $\exists$ pmap $H : Cyl(E) \to F$ s.t., $Hi_0 = f$, and $Hi_1 = g$.

Ref:Lecture 4 inhttps://services.math.duke.edu/~kgw/8803_Stable/

Definition

A right homotopy K from f to g is a morphism $H : A \to \text{path}(B)$ such that the following diagram commutes:



If the right homotopy exists, we write $f \sim_r g$.

(Homotopy is preserved by composition) Let $f, g : A \to B$ be left (resp. right) homotopic. Then, for any $h : A' \to A$ and $k : B \to B'$, kfh is left (resp. right) homotopic to kgh.

When considering morphisms between fibrant or cofibrant objects, the left and right homotopy are nice. (So that is why we need fibrant/cofibrant replacement)

We consider the homotopy relation on $\mathrm{Hom}_{\boldsymbol{M}}(A, B)$ in the following.

- If A is cofibrant and B is fibrant, then the left homotopy and right homotopy **define equivalence relations** on $\mathrm{Hom}_{\boldsymbol{M}}(A, B)$, and the **two relations coincide**.
- **(To what extent the weak equivalence has inverse)** Let A, B be fibrant-cofibrant objects. Then f is **weak equivalence** if and only if f **has a homotopy inverse**.

Weak equivalence in the model category are precisely those inverted by localization functor

Theorem

A morphism in a model category $\boldsymbol{M}$ is weak equivalence if and only if it is inverted by the localization functor $F : \boldsymbol{M} \to \boldsymbol{M}[\mathcal{W}^{-1}]$.

Proof(sketch):

Idea: construct some "tool" categories

Construction

- $\boldsymbol{M}_{cf}$: $\begin{cases} \text{objects are fibrant-cofibrant objects in } \boldsymbol{M} \\ \text{for } x, y \in \boldsymbol{M},\ \mathrm{Hom}_{\boldsymbol{M}_{cf}}(x, y) = \mathrm{Hom}_{\boldsymbol{M}}(x, y) \end{cases}$
- $\boldsymbol{hM_{cf}}$: $\begin{cases} \text{objects are fibrant-cofibrant objects in } \boldsymbol{M} \\ \text{for } x, y \in \boldsymbol{M},\ \mathrm{Hom}_{\boldsymbol{hM_{cf}}}(x, y) = \mathrm{Hom}_{\boldsymbol{M}}(x, y)/\sim \end{cases}$

  where the relation $\sim$ is the equivalence relation defined by left homotopy (or right homotopy).
- $\mathrm{Ho}\,\boldsymbol{M}$: $\begin{cases} \mathrm{ob}(\mathrm{Ho}\,\boldsymbol{M}) = \mathrm{ob}(\boldsymbol{M}) \\ \text{for } x, y \in \boldsymbol{M},\ \mathrm{Hom}_{\mathrm{Ho}\,\boldsymbol{M}}(x, y) = \mathrm{Hom}_{\boldsymbol{M}}(RQx, RQy)/\sim \end{cases}$

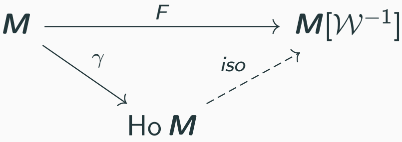
  $\mathrm{Ho}\,\boldsymbol{M}$ is called the homotopy category. The relationship between an abelian category and its derived category parallels the relationship between a model category and its homotopy category.

$$
\begin{array}{ccccc}
\boldsymbol{M} & \xrightarrow{RQ} & \boldsymbol{M}_{cf} & \xrightarrow{\pi} & h\boldsymbol{M}_{cf} \\
& \searrow^{\gamma} & & \nearrow^{\nu} & \\
& & Ho\boldsymbol{M} & &
\end{array}
$$

where γ is bijective on objects and ν is fully faithful. (By definition, it is easy to check its commute)

Show $\boldsymbol{M}[\mathcal{W}^{-1}] \cong \mathrm{Ho}\,\boldsymbol{M}$

- See Theorem 3.4.5 in[Riehl19]. Note that the isomorphism is constructed by factoring F along γ



- The right hand side $\boldsymbol{M}[\mathcal{W}^{-1}]$ does not contain any information of fibrations or cofibrations! Hence, we can only use weak equivalence to describe the homotopy theory (which is known as the relative category). The fibrations and cofibrations are technical ways to do computation.

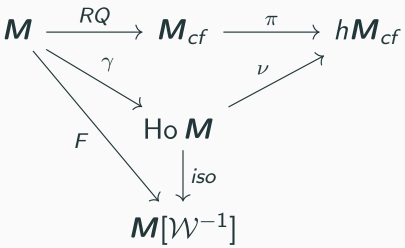
Finally, we show $F : \boldsymbol{M} \to \boldsymbol{M}[\mathcal{W}^{-1}]$, $[f : X \to Y] \mapsto Ff$. Ff is isomorphism if and only if f is weak equivalence.

“ $\Leftrightarrow$” Trivial.

“ $\Rightarrow$” Assume Ff is isomorphism. $\Rightarrow \gamma f$ is isomorphism.

ν is fully faithful $\Rightarrow \nu\gamma f$ is isomorphism

Diagram commutes $\Rightarrow \pi RQf$ is isomorphism



$\boldsymbol{M}_{cf} \xrightarrow{\pi} h\boldsymbol{M}_{cf}$, $RQf \mapsto \overline{RQf}$

$$\begin{cases} RQf \in \mathrm{Hom}_{\boldsymbol{M}}(RQX, RQY) \Rightarrow RQF \text{ is a morphism between fibrant-cofibrant objects} \\ \overline{RQf} \in \mathrm{Hom}_{\boldsymbol{M}}(RQX, RQY)/\sim \text{ is an isomorphism} \Rightarrow RQf \text{ has homotopy inverse} \end{cases}$$

$\Rightarrow RQf$ is a weak equivalence by the above property.

By naturality of $\eta : id \Rightarrow R$ and $\epsilon : Q \Rightarrow id$, we have

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \eta_X}\uparrow & & \uparrow{\scriptstyle \eta_Y} \\ QX & \xrightarrow{Qf} & QY \\ {\scriptstyle \epsilon_{QX}}\downarrow & & \downarrow{\scriptstyle \epsilon_{QY}} \\ RQX & \xrightarrow{RQf} & RQY \end{array}$$

ϵ_{QX}, ϵ_{QY}, RQf are weak equivalence, by two of three, Qf is weak equivalence.

Qf, η_X, η_Y are weak equivalence, by two of three, f is weak equivalence.

Definition

A functor between homotopical categories is a **homotopical functor** if it preserves the classes of weak equivalences.

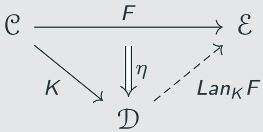
- Homotopical functor is NICE!
- Not all functors are homotopical functors.
- Derived functor is a **universal homotopical approximation** to a given functor. (Analog to CW approximation)

What does universal here mean? The universal is described by **absolute Kan extension**. Unlike ordinary universal property, absolute Kan extension seems to have two kinds of “uniqueness” in the universal property. This is because morphism has only **one way of composing**. However, a natural transformation has two ways to do composition (**Kan extension** shows the natural transformation is “best” in the context of **vertical composition**, and **absolute Kan extension** provides natural transformation is “best” in the context of **horizontal composition**)

Definition

Let $F : \mathcal{C} \to \mathcal{E}$, $K : \mathcal{C} \to \mathcal{D}$ be functors among categories.

A *left Kan extension* of F along K is a pair $(Lan_K F : \mathcal{D} \to \mathcal{E}, \eta : F \Rightarrow Lan_K F \circ K)$, where $Lan_K F$ is a functor and η is a natural transformation, such that for any pair $(G : \mathcal{D} \to \mathcal{E}, \gamma : F \Rightarrow GK)$, there exists a unique natural transformation $\zeta : Lan_K F \Rightarrow G$ such that $\zeta\eta = \gamma$.



Let $F : \mathcal{C} \to \mathcal{E}$, $K : \mathcal{C} \to \mathcal{D}$ be functors among categories.

Definition

A left Kan extension is **absolute** if for any functor $H : \mathcal{E} \to \mathcal{G}$, $(H \circ Lan_K F : \mathcal{D} \to \mathcal{G}, H\eta)$ is a left Kan extension of HF along K.

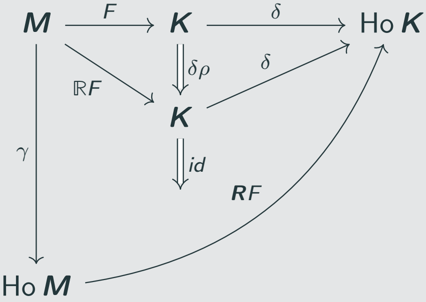
Similarly one can define right Kan extension and absolute right Kan extension.

Since for model category $\boldsymbol{M}[\mathcal{W}^{-1}] \cong \operatorname{Ho} \boldsymbol{M}$, we define homotopy category of a homotopical category $\mathcal{C}$ as $\operatorname{Ho} \mathcal{C} := \mathcal{C}[\mathcal{W}^{-1}]$. So we can write $\iota : \mathcal{C} \to \operatorname{Ho} \mathcal{C}$ be the localization functor.

Let $F : \boldsymbol{M} \to \boldsymbol{K}$ be a functor between homotopical categories. Let $\gamma : \boldsymbol{M} \to \operatorname{Ho} \boldsymbol{M}$, $\delta : \boldsymbol{K} \to \operatorname{Ho} \boldsymbol{K}$ be localization functors.

Definition

A **right derived functor** of F is a homotopical functor $\mathbb{R}F$ together with a natural transformation $\rho : F \Rightarrow \mathbb{R}F$ such that $(\boldsymbol{R}F, id \circ (\delta\rho) = \delta\rho)$ is absolute left Kan extension of δF along γ.



where $\boldsymbol{R}F$ is the universal map of $\text{Ho}\,\boldsymbol{K} \cong \boldsymbol{K}[\mathcal{W}'^{-1}]$, since $\delta\mathbb{R}F$ inverts weak equivalence of $\boldsymbol{M}$ ($\mathbb{R}F$ is homotopical). We refer to $\boldsymbol{R}F : \text{Ho}\,\boldsymbol{M} \to \text{Ho}\,\boldsymbol{K}$ as the *total right derived funtors.*

Remark

The total left (resp. right) derived functor is the functor between homotopy categories induced from given functors between categories by derived left (resp. right) functors.

Quillen adjunction and Quillen equivalence

Property

Let $F : \boldsymbol{M} \leftrightarrows \boldsymbol{K} : G$ be a pair of adjunctions between model categories. TFAE:

- The left adjoint is left Quillen.
- The right adjoint is right Quillen.
- The left adjoint preserves cofibrations, and the right adjoint preserves fibrations.
- The left adjoint preserves trivial cofibrations, and the right adjoint preserves trivial fibrations.

The pair that satisfies one of these conditions is called *Quillen adjunction*.

Theorem

If $F : \boldsymbol{M} \to \boldsymbol{K} : G$ is a Quillen adjunction, then the total left derived functor of the left adjoint functor and total right derived functor of the right adjoint functor forms an adjunction, i.e., $\boldsymbol{L}F : \operatorname{Ho}\boldsymbol{M} \leftrightarrows \operatorname{Ho}\boldsymbol{K} : \boldsymbol{R}G$

"Two model categories present equivalent homotopy theories if there exists a finite sequence of model categories and a zig-zag of Quillen equivalences between them." [**Riehl19**]

Slogan: Quillen equivalence is an equivalence between homotopy categories induced by a Quillen adjunction by derived functors.

Definition

A Quillen adjunction $F : \boldsymbol{M} \rightarrow \boldsymbol{K} : G$ between model categories is a Quillen equivalence if one of the following equivalent conditions holds:

- The total left derived functor of the left adjoint $\boldsymbol{L}F : \mathrm{Ho}\,\boldsymbol{M} \rightarrow \mathrm{Ho}\,\boldsymbol{K}$ is an equivalence of categories.
- The total right derived functor of the right adjoint $\boldsymbol{R}G : \mathrm{Ho}\,\boldsymbol{K} \rightarrow \mathrm{Ho}\,\boldsymbol{M}$ is an equivalence of categories.

Remark

For other equivalence definitions, see Definition 4.5.1 in [**Riehl19**].

Property

There is a Quillen equivalence between the category of simplicial set and the category of topological space: the geometric realization is left adjoint to the functor $Sing(-)$. Hence the homotopy theory of simplicial set is the same as the homotopy theory of topological spaces.

Models for ∞-categories

Note that the “model” here is different from the model category.

Models for ∞-categories (Here ∞-categories means $(\infty, 1)$-categories)

- Quasi-categories (simplicial sets $X_\bullet$ such that $X_\bullet \to *$ has right lifting property to inner horns.)
  - Quasi-categories are one of the most simple models for ∞-category, so in some cases we just refer to quasi-categories as ∞-categories.
  - Homotopy category of quasi-categories have defined at last time
- Category enriched over Kan complexes
  - The property of the Kan complex allows one to find the inverse for any i-morphism in a Kan complex, $i \geqslant 1$, which is the $i+1$-morphism in the Kan complex enriched category.
  - we can define homotopy category of simplicialy enriched categories.

Remark

- Other models of ∞-categories, see [**Bergner09**].
- What's on earth an ∞-category which is independent of models? See [**Infcos**] and chap 9 in [**Riehl19**].

Property

TFAE:

- A simplicially enriched category with $\mathrm{ob}(\mathcal{C})$
- A simplicial object $\mathcal{C}_\bullet : \boldsymbol{\Delta}^{op} \to \mathbf{Cat}$ satisfying: $\mathrm{ob}(\mathcal{C}_n) = \mathrm{ob}(\mathcal{C})$ and each functor $\mathcal{C}_n \to \mathcal{C}_m$ is identity on objects.

We define the homotopy theory as a simplicially enriched category by defining its homotopy category.

Definition

Let $\mathcal{C}$ be a simplicially enriched category; we define the homotopy category $\operatorname{Ho}\mathcal{C}$ consisting of the following data:

- $\operatorname{ob}(\operatorname{Ho}\mathcal{C}) = \operatorname{ob}(\mathcal{C})$
- For $x, y \in \operatorname{ob}(\operatorname{Ho}\mathcal{C})$, let $\operatorname{Hom}_{\operatorname{Ho}\mathcal{C}}(x, y) := \pi_0(\operatorname{Hom}_{\mathcal{C}}(x, y))$ where π_0 is the π_0 of simplicial sets.

Quasi-categories and categories enriched over the Kan complex are already admitted in homotopy theory.

- We have known from model theory that we can obtain homotopy theory.
- What if we introduce model structure on quasi-categories and categories enriched over Kan complexes?

$\Rightarrow$ Homotopy theory of homotopy theory!

Model structure on sSet (and thus on the subcategory of quasi-categories)

- Quillen model structure, $\mathbf{sSet}_{Quillen}$, Example 1.2.13 in [**Maciel**]. Firant objects are Kan complexes
- Joyal model structure, $\mathbf{sSet}_{Joyal}$, Example 1.2.15 in [**Maciel**]. Fibrant objects are all quasi-categories.

Model structure on Cat_{sSet} (and thus on category enriched over Kan complexes)

- Bergner model structure, see Example 1.2.2 in [**Maciel**]. Fibrant objects are categories enriched over Kan complexes.

∞-category that has the same homotopy theory with a given model category

Motivation of finding ∞-categories of model categories.

Definition

A relative category $\mathcal{C}$ is a pair $(und\mathcal{C}, weq\mathcal{C})$, where $und\mathcal{C}$ is a category and $weq\mathcal{C}$ is a wide category. We refer to morphisms in $weq\mathcal{C}$ as weak equivalence in $\mathcal{C}$.

There are many ways to homotopy theory; two are model categories (or, more generally, relative categories) and $(\infty, 1)$-categories.

Given a model category, one can obtain an ∞-category with same homotopy theory as model category.

Hammock localization RelCat $\xrightarrow{L^H}$ Cat$_{sSet}$

Given a model category, we can replace **hom-set** to **simplicial sets** with store homotopical data better. This is the motivation for Hammock localization.

Definition

Let $\mathcal{C}$ be a category together with a subcollection $\mathcal{W}$ of morphisms. We define a simplicially enriched category $L^H(\mathcal{C}, \mathcal{W})$, called the *hammock localization* of $\mathcal{C}$ by $\mathcal{W}$ as following:

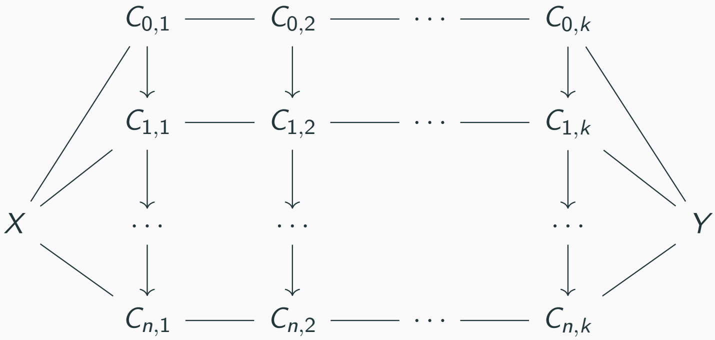
(1)$\text{ob}(L^H(\mathcal{C}, \mathcal{W})) = \text{ob}(\mathcal{C})$

Definition

(2) Let $x, y \in \mathrm{ob}(L^H(\mathcal{C}, \mathcal{W}))$, the set $L^H(\mathcal{C}, \mathcal{W})(x, y)$ is a simplicial set defined as follows:

- $L^H(\mathcal{C}, \mathcal{W})(x, y)_n$ is a collection of elements of the form:
  satisfying the following properties:
  1. all vertical maps lie in $\mathcal{W}$
  2. maps in the same column have the same direction
  3. maps pointing left lie in $\mathcal{W}$
  4. adjacent columns have different directions
  5. every column has a non-identity map

  We call such an element in $L^H(\mathcal{C}, \mathcal{W})(x, y)_n$ a *hammock* of width n and length k
- The face map d^i deletes the i-th row
- The degeneracy map s^i repeats the i-th row


$C_{0,1}$ $C_{0,2}$ $\cdots$ $C_{0,k}$
$C_{1,1}$ $C_{1,2}$ $\cdots$ $C_{1,k}$
X $\cdots$ $\cdots$ $\cdots$ Y
$C_{n,1}$ $C_{n,2}$ $\cdots$ $C_{n,k}$

Fact

A fibrant object in $(\mathbf{Cat}_{sSet})_{Bergner}$ is a category enriched over Kan complex, i.e., an ∞-category.

Hence, a fibrant replacement can obtain an ∞-cateory.

- There exists an Ex^{∞} functor that **replace a simplicial set with a Kan complex**.
- For $\mathcal{C} \in \mathbf{Cat}_{sSet}$, we define

$$\mathbb{R}_B(\mathcal{C}) = \begin{cases} \text{ob}(\mathbb{R}_B(\mathcal{C})) = \text{ob}(\mathcal{C}) \\ \text{for } x, y \in \text{ob}(\mathbb{R}(\mathcal{C})),\ \mathbb{R}(\mathcal{C})(x, y) = Ex^{\infty}\mathcal{C}(x, y) \end{cases}$$

- Here we **do not use** the **natural fibrant replacement functor** as we introduced before. The **advantege** is $\text{ob}(\mathbb{R}_B(\mathcal{C})) = \text{ob}(\mathcal{C})$, we **carries the information of the objects**.

We've obtained a category enriched over the Kan complex. The next step is to obtain a quasi-category from the category enriched by Kan complexes. (Quasi-category is easier: a quasi-category is a simplicial set, but each hom in category enriched by Kan complexed is a simplicial set)

This step we use Homotopy coherent nerve $\mathbf{Cat}_{sSet} \xrightarrow{N^{hc}} \mathbf{sSet}$

We can obtain a model category's underlying ∞-category by applying the functor $ud : N^{hc} \circ \mathcal{M}_B \circ L^H$.

Property

For any model category $\boldsymbol{M}$, $\operatorname{Ho} \boldsymbol{M} \simeq h(ud(\boldsymbol{M}))$.

Remark

- Model categories are 1-category representations of ∞-categories.
- Not all ∞-categories come from model categories; one important condition for ∞-categories to come from model categories is that the ∞-categories have all limits and colimits.

Summary

Let $\boldsymbol{M}$ be a model category.

- Model structure can give a homotopy theory on a general category
- $\operatorname{Ho} \boldsymbol{M} \cong \boldsymbol{M}[\mathcal{W}^{-1}]$
- Weak equivalences in $\boldsymbol{M}$ are precisely morphisms inverted by localization functor $\iota : \boldsymbol{M} \to \boldsymbol{M}[\mathcal{W}^{-1}]$
- We use Quillen equivalence to describe the equivalence of different homotopy theories.
- There is an ∞-category for each model category which has same homotopy theory.

[Bergner09] Julia E Bergner. A survey of $(\infty, 1)$-categories. In Towards higher categories, pages 69–83. Springer, 2009.

[Maciel] Sergio Maciel. The underlying infinity category of a model category. Available at: https://shmaciel.github.io/articles/inf_cat_of_a_model_cat.pdf.

[Infcos] Emily Riehl. infinity-cosmoi. Available at:https://emilyriehl.github.io/infinity-cosmos/.

[Riehl19] Emily Riehl. Homotopical categories: from model categories to $(\infty, 1)$ -categories. arXiv preprint arXiv:1904.00886, 2019.

Thanks!