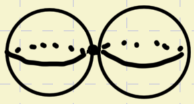
Introduction to intersection homology.

Ref: Singular intersection homology, Greg Friedman

[Exp] Poincare duality $\overset{\text{Corollary}}{\Rightarrow}$ $\dim_{\mathbb{C}} H_i(X;\mathbb{C}) = \dim_{\mathbb{C}} H_{n-i}(X;\mathbb{C})$

$Y =$  $S^2 \vee S^2$

$H_0(Y;\mathbb{C}) = \mathbb{C}$

$H_2(Y;\mathbb{C}) = \mathbb{C} \oplus \mathbb{C}$

$\times$

<u>singular space</u> ⟵ Intersection homology

Outline:

GM intersection homology (NOT right)

- Simplicial intersection homomology ⋆
- PL intersection homology
- singular intersection homology

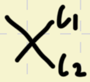
$+$ Non GM intersection homology

• [Def] (filtered space) A <u>filtered space</u> is a Hausdorff topo space X together with a seq. of closed subspaces

$$X = X^n \supseteq X^{n-1} \supseteq \cdots \supseteq X^{-1} = \phi$$

X^i: i-th skeleton; connected component of $X^i - X^{i-1}$: stratum; $X^n - X^{n-1}$: regular stratum; index i: formal dimension; $X^{n-1} =: \Sigma_X$

[Rmk] 1. X^i is always i-dimension singularities

$\mathbb{R}^2$ 

$$X^2 = \mathbb{R}^2 \supseteq X^1 = \underbrace{l_1 \cup l_2}_{\text{1-dim sing.}} \supseteq X^0 = \underbrace{l_1 \cap l_2}_{\text{0-dim sing.}} \supseteq X^{-1} = \phi$$

2. Formal dimension can not equal to topo dim.

e.g (subspace filtration) (Analog to subspace topology)

$Y \subseteq X$, $X^n \supseteq X^{n-1} \supseteq \cdots \supseteq X^0 \supseteq X^{-1} = \phi$

we define $Y^i = X^i \cap Y \rightsquigarrow Y^n \supseteq Y^{n-1} \supseteq \cdots \supseteq Y^{-1} = \phi$.

$X = S^2 \underset{*}{\vee} S^1$. $Y = S^1$. $\quad X = X^2 \supseteq X^1 = S^1 \supseteq X^0 = * \supseteq X^{-1} = \phi$

$\rightsquigarrow$ subspace filtration $\quad Y^2 = S^1 \supseteq Y^1 = S^1 \supseteq Y^0 = * \supseteq Y^{-1} = \phi$

$Y^2 = S^1$ with $\begin{cases} \text{formal dim} & 2 \\ \text{topo dim} & 1 \end{cases}$

不自然的 filtration 是否因为 subspace filtration 本身不合理?

We believe it's a natural way to give subspace filtration for Y.

with subspace filtration, $I^{\bar{p}_Y} S_*^{GM}(Y) \subseteq I^{\bar{p}_X} S_*^{GM}(X)$ is a subcomplex $\rightsquigarrow$ we can define

$$I^{\bar{p}} S_*^{GM}(X) / I^{\bar{p}_X} S_*^{GM}(Y) =: I^{\bar{p}} S_*^{GM}(X, Y)$$

$\rightsquigarrow$ Leading to relative intersection homology

$\square$

- General position. X: simplicial complex.

  i-simplex σ in general position of stratum S if

  $$\dim(\sigma \cap S) \leq \dim(\sigma) + \dim(S) - n$$

  $|\text{simplicial complex}| \simeq \text{manifold} \Rightarrow$ It's possible to (homologous) move σ to be in general position with S

[Exp]

$$X^2 = T^2 \supseteq X^1 = * \supseteq X^0 = * \supseteq X^{-1} = \phi$$

strata: $S_1 = T^2 - *$

$S_2 = *$

σ is an 1-simplex in picture.

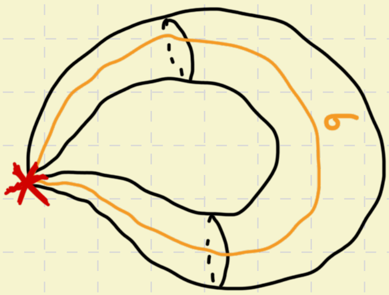
σ in general position with $S_2 = \{*\} \Leftrightarrow$ we have $\dim(\sigma \cap S_2) \leq \dim(\sigma) + \dim(S_2) - n = 1 + 0 - 2 = -1$

$\overset{\dim\phi := -\infty}{\Longleftrightarrow} * \notin \sigma$

we can always move σ not containing $\{*\}$.

But for pinched torus, it's impossible to move σ not containing $*$!

(即不存在等价类的代表元是不包含$*$的)



$\square$

在上个例子中, pinched torus 里的 σ 探测到了空间中的奇点 $*$.

在多大程度上容忍这种怪异的 simplex 可以反映 singular space 的信息.

我们在条件 $\dim(\sigma \cap S) \leq \dim(\sigma) + \dim(S) - n$ 的右手边加 调节项 $\bar{p}(S)$ 来刻画容忍程度.

[Def] (Perversity) X : filtered sp of formal dim n

$\mathcal{F} = \{$ strata of $X\}$. A perversity on X is a function

$\bar{p} : \mathcal{F} \to \mathbb{Z}$ s.t. $\bar{p}(S) = 0$ if $S \subset X - \Sigma_X$,

i.e., if S is a regular stratum $\square$

[Rmk] Why $\bar{p}(S)=0$ is clear when we consider definition of $\bar{p}$-allowable simplexes. $\square$

[Def] X: simplicial filtered sp with perversity $\bar{p}$

$C_*(X)$: chain complex of X

i-simplex σ is called $\bar{p}$-allowable if

$$\underbrace{\dim(\sigma \cap S)}_{\text{topo dim}} \leq i - \underbrace{\operatorname{codim}(S)}_{\text{formal dim.}} + \bar{p}(S), \ \forall \text{ stratum } S \text{ of } X$$ $\square$

$\square$

[Def] A chain $\xi \in C_i(X)$ is <u>$\bar{p}$-allowable</u> if $\forall$ simplices of ξ and $\partial\xi$ are $\bar{p}$-allowable.

$I^{\bar{p}}C_*(X) = \{\xi \in C_i(X) \mid \xi$ is $\bar{p}$-allowable$\}$

[Rmk] $\xi \in I^{\bar{p}}C_i(X)$, $\forall$ simplex in $\partial\xi$ and $\partial^2\xi = 0$ are $\bar{p}$-allowable.

So $\partial\xi \in I^{\bar{p}}C_i(X)$. So $(C_*(X), \partial)$ restricts to chain complex $(I^{\bar{p}}C_*(X), \partial)$

[Def] $I^{\bar{p}}H_*^{GM}(X) := H_*(I^{\bar{p}}C_*^{GM}(X))$

[Rmk] We come back to the question: Why $\bar{p}(S)=0$ for regular stratum S?

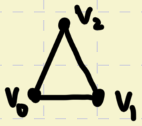
Let S_R be any regular stratum. The condition of simplex σ being $\bar{p}$-allowable with S_R is: $\dim(\sigma \cap S_R) \leq i - \underbrace{\operatorname{codim} S_R}_{=0} + \underbrace{\bar{p}(S_R)}_{=0} = i - 0 + 0 = i$.

Always holds !!!

($\bar{p}(S)=0$ means we want regular stratum be vaccum for simplex.)

下面是若干 simplicial intersection homology 的例子.

[Exp1] perversity 对于 intersection homology 的调控未必是敏感的.
当 $\bar{p}(v_0)$ 变化的时候, intersection homology 只有三类结果.

$X=$  is the boundary of a 2-simlex , $\underset{X}{\overset{\parallel}{X^1}} \supseteq \underset{\{v_0\}}{\overset{\parallel}{X^0}} \supseteq \underset{\emptyset}{\overset{\parallel}{X^{-1}}}$.

For $\forall$ 0-simplex v

$$\dim(v\cap\{v_0\}) \leq \dim v+\dim v_0 - n + \bar{p}(v_0)$$
$$= 0+0-1+\bar{p}(v_0) = \bar{p}(v_0)-1$$

$\bar{p}$-allowable 0-simplex $\begin{cases} v_0, v_1, v_2 & \bar{p}(v_0)\geqslant 1 \\ v_1, v_2 & \bar{p}(v_0)<1 \end{cases}$

For $\forall$ 1-simplex e

$$\dim(e\cap v_0) \leq \dim e + \dim v_0 - n + \bar{p}(v_0) = 1+0-1+\bar{p}(v_0) = \bar{p}(v_0)$$

$\bar{p}$ allowable 1-simplex $\begin{cases} [v_1,v_2], [v_0,v_1], [v_0,v_2] & \bar{p}(v_0)\geqslant 0 \\ [v_1,v_2] & \bar{p}(v_0)<0 \end{cases}$

(i) $\bar{p}(v_0)\geqslant 1$ $\quad I^{\bar{p}}H_*(X) = H_*(X)$

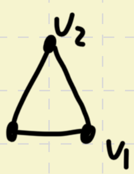
(ii) $\bar{p}(v_0)<0$ $\quad I^{\bar{p}}H_*(X) = H_*($  $)$

(iii) $\bar{p}(v_0)=0$ $\quad I^{\bar{p}}H_0(X)=\mathbb{Z}$

$I^{\bar{p}}H_1(X)=\mathbb{Z}$ $\quad (I^{\bar{p}}C_*(X) \subseteq C_*(X))$

cycles in $I^{\bar{p}}C_*(X)$ comes from $C_*(X)$

[Exp] Filtration impacts intersection homology

$X=$ 

$$X = X^1 \supseteq X^0 = \{v_0, v_1, v_2\} \supseteq X^{-1}$$

singular stratum : v_0, v_1, v_2 .

$\bar{p}(v_i)=0$

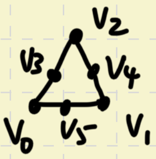
$\forall$ 0-simplex v, $\dim(v\cap v_i) \leq 0+0-1+\underset{\overset{\parallel}{0}}{\bar{p}(v_i)} = -1$

$\Rightarrow$ all 0-simplex not allowable

$\forall$ 1-simplex e, $\dim(e \cap V_i) \leq 1+0-1+\bar{p}(V_i) = 0$ always holds

So $I^{\bar{p}}H_0(X) = 0$, $I^{\bar{p}}H_1(X) = \mathbb{Z}$

[Exp] Subdivision impacts intersection homology

$X=$ 

$X = X^1 \supseteq X^0 = \{v_0, v_1, v_2\} \supseteq X^{-1} = \phi$

$\bar{p}(v_i) = 0$

$\dim(v \cap V_i) \leq -1 \rightsquigarrow$ allowable 0-simplexes: v_3, v_4, v_5

$\dim(e \cap V_i) \leq 0 \rightsquigarrow$ all 1-simplexes are allowable.

$I^{\bar{p}}H_1(X) = \mathbb{Z}$, $I^{\bar{p}}H_0(X) = \mathbb{Z}$

[Rmk] 个人理解：perversity 和同调 degree n 很相像，需要计算越多越好，并不是某一个 perversity 是最好的 perversity，只计算那一个 perversity. 事实上，可以定义 dual perversity：

More generally, if X is any R-oriented locally $(\bar{p}; R)$-torsion-free n-dimensional stratified pseudomanifold, we have a Poincaré duality isomorphism

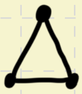
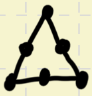
$$\mathcal{D} : I_{\bar{p}}H^i_c(X; R) \to I^{D\bar{p}}H_{n-i}(X; R),$$

不仅 degree 上有对偶，perversity 上也有对偶. □

PL Intersection homology

我们先定义 PL homology，再定义 PL intersection homology.

Recall simplicial complex by example:

[Exp] $K_1 =$  , $K_2 =$  (are simplicial complex.)

$|K_1| = |K_2| = \triangle$ is topo space $\square$

[Def] The simplicial complex K' is a <u>subdivision</u> of K if (i) $|K| = |K'|$

(ii) $\forall$ simplex of $K' \subseteq$ in some simplex of K.

(Exp)

[Def] (Triangulation) A <u>triangulation</u> T of a topo sp X is a pair $T = (K, h)$

K: locally finite simplicial complex

$h: |K| \to X$ be a homeomorphism.

* locally finite: $\forall x \in |K|$, $\exists$ n.b.h. U intersects finite number of simplexes

[Def] Let $T = (K, h)$, $S = (L, j)$ be two triangulations.

$T = (K, h) \sim S = (L, j) \Leftrightarrow j^{-1}h$ is simplicial iso

[Def] (PL space) A PL (piecewise linear) space is a topo sp X with $\mathcal{T} = \{$ locally finite triangulations $\}$ s.t.

(i) $\forall T \in \mathcal{T}$, subdivision of T contained in $\mathcal{T}$

(ii) $\forall T, S \in \mathcal{T}$, T, S has <u>common refinement.</u>

* $T = (K, k)$, $S = (L, l)$. $\exists$ subdivison $T' = (K', k)$ of T, $\exists$ subdivision $S' = (L', l)$ of S. s.t. $l^{-1}k: K' \to L'$ iso.

[Construction] $T = (K, k)$, $S = (L, l) \in \mathcal{T}$

$T \leq S \Leftrightarrow S$ equiv. to a subdivision of T

[Fact] $(\mathcal{T}, \leq)$ is a directed set.

[Def] $\mathcal{C}_*(X) = \varinjlim_{T \in \mathcal{T}} C_*^T(X)$, where $C_*^T(X) = C_*(K)$ for $T = (K, k)$

[Rmk] ① $\varinjlim_{T \in \mathcal{T}} C_*^T(X)$ has concrete construction

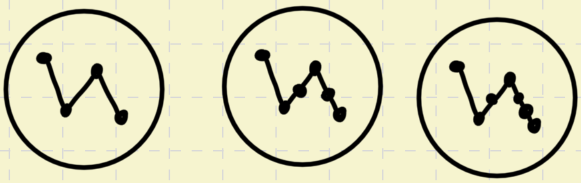
$$\varinjlim_{T \in \mathcal{T}} C_*^T(X) = \bigcup_{T \in \mathcal{T}} C_*^T(X) \Big/ \sim \quad \text{where } \xi \sim \eta \Leftrightarrow \xi \text{ and } \eta \text{ maps to same image in } \varinjlim_{T \in \mathcal{T}} C_*^T(X)$$

$$\left(\begin{tikzcd} \xi \in C_*^T(X) \arrow[dr] & & \eta \in C_*^S(X) \arrow[dl] \\ & \varinjlim_{T \in \mathcal{T}} C_*^T(X) & \end{tikzcd}\right)$$

Let $[\xi]$ denote the equiv. class.

② $[\xi] = [\eta]$ iff their image agree in some common subdivision.

e.g.



are same elements in $\mathcal{C}_*(X)$

③ X is a PL space with admissible triangulations $\mathcal{T}$.

Let $T_0 = (K, h) \in \mathcal{T}$ and let $\mathcal{T}_0 = \{T \in \mathcal{T} \mid T \text{ subdivision of } T_0\}$

Then $$\mathcal{C}_*(X) = \varinjlim_{T \in \mathcal{T}} C_*^T(X) \cong \varinjlim_{T \in \mathcal{T}_0} C_*^T(X)$$

[Def] X : PL space. Define $\Omega_*(X) := H_*(\mathcal{C}_*(X))$

[Def] X : PL filtered sp s.t. $\forall$ skeleton X^i is a subcomplex of any admissible triangulation.

Define $I^{\bar{p}}\mathcal{C}_*(X) = \varinjlim_{T \in \mathcal{T}} I^{\bar{p}} C_*^{GM,T}(X)$, where $I^{\bar{p}} C_*^{GM,T}(X) := I^{\bar{p}} C_*^{GM}(|K|)$

[Rmk] : Skeleton can inherit triangulation from X w.r.t. any admissible triangulation.

[Rmk] filtration & perversity of X can "move to" $|K|$ by homeo k.

[Fact] $T \le T'$ subdivision chain map $\bar{\nu}: C_*^T(X) \to C_*^{T'}(X)$ restricts to a map $\nu: I^{\bar{p}}C_*^{GM,T}(X) \to I^{\bar{p}}C_*^{GM,T'}(X)$

[Def] $I^{\bar{p}}\mathfrak{H}_*^{GM}(X) = H_*(I^{\bar{p}}\mathfrak{C}_*^{GM}(X)) \cong \varinjlim_{T\in\mathcal{T}} H_*(I^{\bar{p}}C_*^{GM,T}(X))$

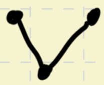
$$= \varinjlim_{T\in\mathcal{T}} I^{\bar{p}}H_*^{GM,T}(X)$$

[Prop] Let $\xi \in \mathfrak{C}_i(X)$.

$$\xi \in I^{\bar{p}}\mathfrak{C}_i(X) \Leftrightarrow \begin{cases} \dim(|\xi| \cap S) \le i - \operatorname{codim} S + \bar{p}(S) \\ \dim(|\partial\xi| \cap S) \le i-1-\operatorname{codim} S + \bar{p}(S) \end{cases} \quad \text{for } \forall \text{ stratum } S \text{ of } X$$

[Def] $L \subseteq K$. L is called full subcomplex if

$\forall \sigma \in K$ with vertices in $L \Rightarrow \sigma \in L$

[Exp] $K =$  , $L =$  L is NOT full subcomplex of K.

[Def] (Full triangulation) An admissible triangulation T of PL filtered space X is called <u>full triangulation</u> if

$\forall X^i$ is full subcomplex of X.

[Thm] X: PL filtered space.

T: full triangulation.

T': any subdivision of T.

Then $I^{\bar{p}}C_*^{GM,T} \longrightarrow I^{\bar{p}}C_*^{GM,T'}(X)$ is an iso

[Coro] $I^{\bar{p}}\mathfrak{H}_*^{GM}(X) = H_*(I^{\bar{p}}\mathfrak{C}_*^{GM}(X))$

$$= H_*(\varinjlim_{T\in\mathcal{T}_0} I^{\bar{p}}C_*^{GM,T}(X))$$

$$\cong H_*(I^{\bar{p}}C_*^{GM,T}(X)) = I^{\bar{p}}H_*^{GM,T}(X)$$

(No example for computing PL intersection homology. 只要取一个 full triangulation, 就回到 simplicial intersection homology)

Singular homology

[Def] X : filtered space with general perversity $\bar{p}$

$S_*(X)$: Singular chain complex of X, i.e., $S_i(X) = \{\Delta^i \to X\}$

A singular i-simplex $\sigma : \Delta^i \to X$ is called $\bar{p}$-allowable if

$\sigma^{-1}(S) \subseteq \{(i - \operatorname{codim}(S) + \bar{p}(S)) - \text{skeleton of } \Delta^i\}$ for all strata S of X.

A chain $\xi \in S_i(X)$ is $\bar{p}$-allowable if all of the simplices in ξ and all of the simplices of $\partial\xi$ are $\bar{p}$-allowable.

Let $I^{\bar{p}} S_*^{GM}(X) = \{\xi \in S_*(X) \mid \xi \text{ is } \bar{p}\text{-allowable}\}$

Define singular intersection homology $I^{\bar{p}} H_*^{GM}(X) = H_*(I^{\bar{p}} S_*^{GM}(X))$

[Exp] (Singular homology 可以和 simplicial homology 相同)

X is a simplicial filtered space, and the singular simplex $\sigma \hookrightarrow X$ is inclusion. 则 $\sigma^{-1}(S) = \sigma \cap S$. 故 $\dim(\sigma^{-1}(S)) \leq i - \operatorname{codim}(S) + \bar{p}(S)$

等价于 $\dim(\sigma \cap S) \leq i - \operatorname{codim}(S) + \bar{p}(S)$

(Singular homology is not easy to compute by hand, so there is no more appropriate examples)

Theorem 5.4.2 *Let X be a PL filtered space with triangulation T, and let $W \subset X$ be an open subset of X such that W is a PL CS set. Then the composition*

$$I^{\bar{p}}\mathfrak{H}_*^{\mathrm{GM}}(W;G) \xrightarrow{\theta^{-1}} I^{\bar{p}}\mathfrak{H}_*^{\mathrm{GM},T}(W;G) \xrightarrow{\psi} H_*(I^{\bar{p}}\mathfrak{S}_*^{\mathrm{GM}}(W;G))$$

is an isomorphism. In particular, $I^{\bar{p}}\mathfrak{H}_*^{\mathrm{GM}}(W;G) \cong I^{\bar{p}}H_*^{\mathrm{GM}}(W;G)$, and if X is a PL CS set then $I^{\bar{p}}\mathfrak{H}_*^{\mathrm{GM}}(X;G) \cong I^{\bar{p}}H_*^{\mathrm{GM}}(X;G)$.

Big picture

Relationship: simplicial $\underset{\text{full triangulation}}{=}$ PL $\underset{\text{"some triangulation"}}{=}$ Singular
(Thm 5.4.2 in Ref)

对于 Intersection homology，在对普通的条件进行修正后，会得到平行的结论 .

e.g.

ordinary homology

$f: X \to Y$ is a homotopy equiv, then $f_*: H_*(X) \xrightarrow{\cong} H_*(Y)$

PL intersection homology

$f: X\to Y$ is a stratified homotopy equiv and $\bar{p}_X(S) = \bar{q}_Y(T)$ if $f(S)\subseteq T$. Then f induces

$$I^{\bar p}H_*^{GM}(X) \cong I^{\bar q}H_*^{GM}(Y)$$

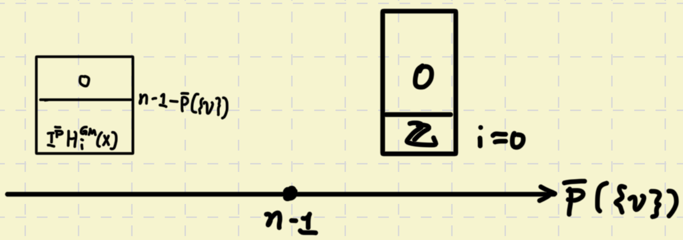
[Rmk] For more props, see ch 4&5 in Ref.

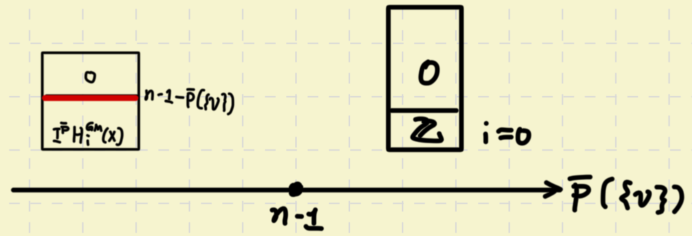
[Rmk] 和 ordinary singular homology 类似, singular intersection homology is not easy to compute by hand. 因此计算 singular intersection homology 时需要使用 tools like "Intersection version of Mayer seq" see Ch 5 in Ref.

Non GM intersection homology

[Exp] X : compact $(n-1)$-dimensional filtered space and assume X has regular strata (so $\exists$ allowable 0-simplex s.t. $I^{\bar p}H_0^{GM}(X)\neq 0$)

$$I^{\bar p}H_i^{GM}(cX) \cong \begin{cases} 0 & i \geq n-\bar p(\{v\})-1,\ i\neq 0 \\ \mathbb{Z} & i=0\geq n-\bar p(\{v\})-1 \\ I^{\bar p}H_i^{GM}(X) & i< n-\bar p(\{v\})-1 \end{cases}$$





当 $\bar p(\{v\}) < n-1$ 时，以 $i = n-1-\bar p(\{v\})$ 为界，$i \geq n-1-\bar p(\{v\})$ $I^{\bar p}H_i = 0$，
$i < n-1-\bar p(\{v\}) = I^{\bar p}H_i^{GM}(X)$. 此时随着 $\bar p(\{v\})$ 增大，会越早出现 0 .
在极限情形下，即 $\bar p(\{v\}) = n-1$ 时，应当有从 0 阶开始所有同调群都是 0.
但事实是 0 阶同调群是 $\mathbb{Z}$.

It suggests that GM intersection homology done well for "small" $\bar p$, but not right for "large" $\bar p$!

[Exp] This example show you why GM intersection homology is not "right" homology theory.

M : n-dim ∂-mf with $\partial M \neq \varphi$.
$M^+ := M \cup_{\partial M} \mathring{c}(\partial M)$ with cone pt v.

$$I^{\bar p}H_i^{GM}(M^+) \cong \begin{cases} H_i(M, \partial M) & i > n-\bar p(\{v\})-1 \\ \mathrm{Im}(H_i(M) \to H_i(M,\partial M)) & i = n-\bar p(\{v\})-1 \\ H_i(M) & i < n-\bar p(\{v\})-1 \end{cases}$$

$$\frac{\text{relative homology grp}}{\text{absolute homology grp}} \quad i = n-\bar p(\{v\})-1$$

当 $\bar p(\{v\})$ 足够大，则我们期待在 degree 0 处看到 relative homology behavior
但这不符合事实.

[Idea] 尝试引入 Non GM intersection homology

- Behavior more like relative group
- Agree with GM intersection homology for small perversity.

改造 singular chain complex, we hope it behavior as relative singular chain complex $S_*(X, \Sigma; G)$ with coeffient G . 因此，落在 Σ 中的 simplex 需要扔掉，因为它在 $S_*(X,\Sigma;G)$ 中是 0.

[Def] Let $S_i^{\bar{p}}(X;G) \subseteq S_i(X;G)$ generated by $\bar{p}$-allowable i-simplex σ with support $|\sigma| \not\subset |\Sigma|$.

[Def] $\hat{\partial}\sigma = \sum_{|\sigma_j| \not\subset \Sigma_X} (-1)^j \sigma_j$

[Rmk] $\hat{\partial}\sigma$ is obtained from $\partial\sigma$ by throwing out the simplices with image in Σ.

[Def] Let $I^{\bar{p}}S_i(X;G) = \{ \xi \in S_i^{\bar{p}}(X;G) \mid \hat{\partial}\xi \in S_{i-1}^{\bar{p}}(X;G) \}$.

$(I^{\bar{p}}S_i(X;G), \hat{\partial})$ is a chain complex, and then we define non-GM intersection homology $I^{\bar{p}}H_*(X;G) = H_*(I^{\bar{p}}S_*(X;G))$.

[Rmk] 对 simplicial 与 PL intersection homology 有类似的定义.