Homotopy limit and colimit I

Zhou Fang

May 27, 2025

Shenzhen Institute for Quantum Science and Engineering,
Southern University of Science and Technology

To compute intersection homotopy, we need to compute the homotopy pushout, which is a homotopy colimit.

From derived functor to homotopy limit and colimit

Definition

A functor between homotopical categories is a **homotopical functor** if it preserves the classes of weak equivalences.

- Homotopical functor is NICE!
- Not all functors are homotopical functors.
- Derived functor is a **universal homotopical approximation** to a given functor. (Analog to CW approximation)

What does universal here mean?

The universal is described by **absolute Kan extension**.

- Unlike ordinary universal property, absolute Kan extension seems to have two kinds of “uniqueness” (different from ordinary universal property)
- This comes from that a natural transformation has two ways to do composition (vertical composition and horizontal composition), but the ordinary morphism only has one way of composing.

Definition

Let $F : \mathcal{C} \to \mathcal{E}$, $K : \mathcal{C} \to \mathcal{D}$ be functors among categories.

A **left Kan extension** of F along K is a pair $(Lan_K F : \mathcal{D} \to \mathcal{E}, \eta : F \Rightarrow Lan_K F \circ K)$, where $Lan_K F$ is a functor and η is a natural transformation, such that for any pair $(G : \mathcal{D} \to \mathcal{E}, \gamma : F \Rightarrow GK)$, there exists a unique natural transformation $\zeta : Lan_K F \Rightarrow G$ such that $\zeta\eta = \gamma$.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "Lan_K F"'] & \end{tikzcd}$$

Kan extension shows the natural transformation is “best” in the context of **vertical composition**

Definition

A left Kan extension is **absolute** if for any functor $H : \mathcal{E} \to \mathcal{G}$, $(H \circ Lan_K F : \mathcal{D} \to \mathcal{G}, H\eta)$ is a left Kan extension of HF along K.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & & \mathcal{E} \arrow[r, "H"] & \mathcal{G} \\
& \mathcal{D} \arrow[ur, "Lan_K F"'] & &
\end{tikzcd}$$

absolute Kan extension provides natural transformation is “best” in the context of **horizontal composition**

Similarly one can define right Kan extension and absolute right Kan extension.

Since for model category $\boldsymbol{M}[\mathcal{W}^{-1}] \cong \operatorname{Ho} \boldsymbol{M}$, we define homotopy category of a homotopical category $\mathcal{C}$ as $\operatorname{Ho} \mathcal{C} := \mathcal{C}[\mathcal{W}^{-1}]$. So we can write $\iota : \mathcal{C} \to \operatorname{Ho} \mathcal{C}$ be the localization functor.

Let $F : \boldsymbol{M} \to \boldsymbol{K}$ be a functor between homotopical categories. Let $\gamma : \boldsymbol{M} \to \operatorname{Ho} \boldsymbol{M}$, $\delta : \boldsymbol{K} \to \operatorname{Ho} \boldsymbol{K}$ be localization functors.

Definition

A **right derived functor** of F is a homotopical functor $\mathbb{R}F$ together with a natural transformation $\rho : F \Rightarrow \mathbb{R}F$ such that $(\boldsymbol{R}F, id \circ (\delta\rho) = \delta\rho)$ is absolute left Kan extension of δF along γ.

$$\begin{tikzcd}
\boldsymbol{M} \arrow[r, "F"] \arrow[dr, "\mathbb{R}F"'] \arrow[dd, "\gamma"'] & \boldsymbol{K} \arrow[r, "\delta"] \arrow[d, phantom, "\Downarrow \delta\rho"] & \mathrm{Ho}\,\boldsymbol{K} \\
& \boldsymbol{K} \arrow[ur, "\delta"] \arrow[d, phantom, "\Downarrow id"] & \\
\mathrm{Ho}\,\boldsymbol{M} \arrow[uurr, bend right, "\boldsymbol{R}F"'] & &
\end{tikzcd}$$

where $\boldsymbol{R}F$ is the universal map of $\mathrm{Ho}\,\boldsymbol{K} \cong \boldsymbol{K}[\mathcal{W}'^{-1}]$, since $\delta\mathbb{R}F$ inverts weak equivalence of $\boldsymbol{M}$ ($\mathbb{R}F$ is homotopical). We refer to $\boldsymbol{R}F : \mathrm{Ho}\,\boldsymbol{M} \to \mathrm{Ho}\,\boldsymbol{K}$ as the *total right derived funtors*.

More simply the diagram can be drawn as

$$\begin{tikzcd}
\boldsymbol{M} \arrow[r, "F"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\Downarrow \delta\rho"] & \boldsymbol{K} \arrow[d, "\delta"] \\
\mathrm{Ho}\, \boldsymbol{M} \arrow[r, "\boldsymbol{R}F"] & \mathrm{Ho}\, \boldsymbol{K}
\end{tikzcd}$$

Remark

The total left (resp. right) derived functor is the universal functor (absolute Kan extension) between homotopy categories induced by derived left (resp. right) functors.

Definition

Let $F : \boldsymbol{M} \to \boldsymbol{K}$ be a functor between model categories. F is *left Quillen* if it preserves cofibrations, trivial cofibrations, and cofibrant objects. F is *right Quillen* if it preserves fibrations, trivial fibrations, and fibration objects.

Property

Let $F : \boldsymbol{M} \leftrightarrows \boldsymbol{K} : G$ be a pair of adjunctions between model categories. TFAE:

- The left adjoint is left Quillen.
- The right adjoint is right Quillen.
- The left adjoint preserves cofibrations, and the right adjoint preserves fibrations.
- The left adjoint preserves trivial cofibrations, and the right adjoint preserves trivial fibrations.

The pair that satisfies one of these conditions is called *Quillen adjunction*.

Total derived funtor preserves Quillen adjunction.

Theorem

If $F : \boldsymbol{M} \to \boldsymbol{K} : G$ is a Quillen adjunction, then the total left derived functor of the left adjoint functor $\boldsymbol{L}F$ and total right derived functor of the right adjoint functor $\boldsymbol{R}G$ forms an adjunction, i.e., $\boldsymbol{L}F : \operatorname{Ho}\boldsymbol{M} \leftrightarrows \operatorname{Ho}\boldsymbol{K} : \boldsymbol{R}G$

There is a useful way to create left/right derived functors, using left/right deformation as following.

Definition

A *left deformation* of model category $\boldsymbol{M}$ provides a natural transformation which is weak equivalence at each level from an endofuntor E to the identity functor, i.e., a natural weak equivalence $q : E \Rightarrow \mathrm{id}$

Remark

- An example of left deformation is the cofibrant replacement Q in a model category.
- The endofunctor E in the left deformation must be homotopical. By naturality, we have

$$\begin{tikzcd} EX \arrow[r, "q_X"] \arrow[d, "Ef"'] & X \arrow[d, "f"] \\ EY \arrow[r, "q_Y"] & Y \end{tikzcd}$$

q_X, q_Y are weak equivalence, if f is weak equivalence, then Ef is weak equivalence by two-of-three property.

Construction

Denote $\mathcal{C}_c$ be any full subcategory of $\mathcal{C}$ **containing** (not necessarily equal) the image of functor E. We call $\mathcal{C}_c$ the subcategory of cofibrant objects.

Remark

It has **no relation** with cofibrant objects defined in the frame of Quillen's model structure. (A confusing name!)

Definition

Let $F : \mathcal{C} \to \mathcal{D}$ be functors between homotopical categories. We say F is *left deformable* if there exists a left deformation on $\mathcal{C}$ and a subcategory of cofibrant objects $\mathcal{C}_c$ s.t. F is homotopical restrict to $\mathcal{C}_c$.

F left deformable $\Rightarrow$ F is homotopical on a very nice subcategory $\mathcal{C}_c$

So it's reasonable that we can "deform" the homotopical functor over subcategory to the homotopical functor over whole category, which is the derived fuctor (derived functor is the universal homotopical functor approximation)

Theorem

Let $F : \mathcal{C} \to \mathcal{D}$ be a left deformable functor corresponding to the left deformation $q : E \Rightarrow \mathrm{id}$ of $\mathcal{C}$. Then FE is a left derived functor of F.

Fact

Let $F : \boldsymbol{M} \to \boldsymbol{K}$ be a functor from a model category to a category with a class of weak equivalences satisfying the two-of-three property. If F carries trivial fibrations in $\boldsymbol{M}$ to weak equivalences in $\boldsymbol{K}$, then F carries all weak equivalences between fibrant objects in $\boldsymbol{M}$ to weak equivalences in $\boldsymbol{K}$.

Property

The left derived functor of any left Quillen functor F (between model category) is FQ, where Q is the cofibrant replacement functor. The right derived functor of any right Quillen functor G (between model category) is GR, where R is the fibrant replacement functor.

Proof.

We only show $G : \boldsymbol{K} \to \boldsymbol{M}$ has right deformation $\eta : id \Rightarrow R$. We w.t.s. G is homotopical on the full subcategory spanned by image of R, denoted by $\boldsymbol{K}_C$. G **carries trivial fibrations to trivial fibrations** since G is right Quillen. So **by the fact** G carries all weak equivalence $RX \xrightarrow{\sim} RY$ to weak equivalences, leading to G is right deformable. So by property GR is the right derived functor. $\square$

Here are two pushouts. We have $S^{n-1} = S^{n-1}$, $D^n \sim *$, but the pushout does not satisfying $S^n \sim *$!

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & D^n \arrow[d] \\ D^n \arrow[r] & S^n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

"The pushout functor fails to preserves component wise homotopy equivalences.

When the functor fails to be homotopical, the next best option is to replace it by a derived functor."[**Riehl19**].

Notation: $\mathcal{C}^{\mathcal{D}} := \mathrm{Fun}(\mathcal{D}, \mathcal{C})$.

Construction

(**limit and colimit functor**) Let $\mathcal{C}$ be a category, and $\mathcal{D}$ be a small category. We can define a colimit functor colim : $\mathcal{C}^{\mathcal{D}} \to \mathcal{C}$ as $F \mapsto \operatorname{colim} F$. Similarly we can define limit funcor lim : $\mathcal{C}^{\mathcal{D}} \to \mathcal{C}$, $F \mapsto \lim F$.

colim : $\boldsymbol{M}^{\mathcal{D}} \to \boldsymbol{M}$ is a functor. For $\alpha : F \Rightarrow G$, there is a unique map factor through colim F of $F(X \xrightarrow{\alpha_X}) G(X) \to \operatorname{colim} G$, which is the map l in the following diagram.

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_x"] \arrow[dr] & G(X) \arrow[r] & \operatorname{colim} G \\
& \operatorname{colim} F \arrow[ur, dashed, "l"] &
\end{tikzcd}$$

We define $\operatorname{colim}(\alpha) = l$.

Definition

Let $\boldsymbol{M}$ be a homotopical category and $\mathcal{D}$ be a small category. The homotopy colimit functor (if it exists), is a left derived functor $\mathbb{L}\,\mathrm{colim} : \boldsymbol{M}^{\mathcal{D}} \to \boldsymbol{M}$ and the homotopy limit functor (if it exists), is a right derived functor $\mathbb{R}\lim : \boldsymbol{M}^{\mathcal{D}} \to \boldsymbol{M}$

Construction

(**Diagonal functor**) Let $\Delta : \boldsymbol{M} \to \boldsymbol{M}^{\mathcal{D}}$ be the functor, where for any object $x \in \boldsymbol{M}$, $\Delta(x)$ is the constant diagram $F(y) = x$ and $F(f) = id_x$ for any $x \in \mathcal{D}$ and $f \in \mathrm{Mor}(\mathcal{D})$.

Remark

colimit are left adjoints to diagonal functor. Hence it's reasonable to pick left derived functor of colimit functor to obtain homotopy colimit functor.

Unfortunately, homotopy colimit and homotopy limit are not easily computed as we want. But here is still a way:

Construction

(Model structure on $\boldsymbol{M}^{\mathcal{D}}$) Let $\boldsymbol{M}$ is a model category and $\mathcal{D}$ be a small category.

1. The projective model structure on $\boldsymbol{M}^{\mathcal{D}}$ is defined as: weak equivalence and fibrations defined point wise in $\boldsymbol{M}$
2. The injective model structure on $\boldsymbol{M}^{\mathcal{D}}$ is defined as following: weak equivalence and cofibrations defined point wise in $\boldsymbol{M}$.

Theorem

Let $\boldsymbol{M}$ is a model category and $\mathcal{D}$ be a small category.

- If the projective model structure on $\boldsymbol{M}^{\mathcal{D}}$ exists then the homotopy colimit $\mathbb{L}\,\mathrm{colim} : \boldsymbol{M}^{\mathcal{D}} \to \boldsymbol{M}$ exists and can be computed by the colimit of a projective cofibrant replacement of the origianl diagram.
- If the Injective model structure on $\boldsymbol{M}^{\mathcal{D}}$ exists then the homotopy limit $\mathbb{R}\,\mathrm{colim} : \boldsymbol{M}^{\mathcal{D}} \to \boldsymbol{M}$ exists and can be computed by the limit of a injective fibrant replacement of the origianl diagram.

proof We only prove (1). Assume $\boldsymbol{M}^{\mathcal{D}}$ has projective model structure.

Claim: The functor $\mathrm{colim} : \boldsymbol{M}^{\mathcal{D}} \to \boldsymbol{M}$ is left Quillen.

Accept the claim, the left derived functor $\mathbb{L}\,\mathrm{colim}$ of left Quillen functor colim exists and is given by $\mathrm{colim} \circ Q$ where Q is the cofibrant replacement in the projective model.

proof of the claim: $\mathrm{colim} : \boldsymbol{M}^{\mathcal{D}} \leftrightarrows \boldsymbol{M} : \Delta$ is an **adjunction**, where Δ is the diagonal functor. To show colim is left Quillen, **it's equivalent to show Δ is right Quillen.** (Which is more easier!)

It suffices to show Δ preserves fibrations, trivial fibrations, fibrants.

$$\Delta : \boldsymbol{M} \longrightarrow \boldsymbol{M}^{\mathcal{D}}$$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, two heads, "f"] & F \arrow[d, Rightarrow, "\alpha"] \\
Y \arrow[r, mapsto] & G
\end{tikzcd}$$

where F and G are functors with $F(d) = x$, $F(f) = id_x$; $G(d) = y$, $G(f) = id_y$.

- $[\alpha_d : F(d) \to G(d)] = [f : x \to y]$ (by definition) is a fibration. By definition of projective model structure, α **is fibration**.
- Similarly one can show Δ **preserves weak equivalences**.
- $\boldsymbol{M}$ has terminal object, and Δ as a right adjoint preserves terminal object. For any $X \twoheadrightarrow *$, we have $\Delta(X) \twoheadrightarrow \Delta(*)$ shows $\Delta(X)$ is fibrant.

Hence Δ is right Quillen.

Fact

The diagram consists of two cofibrations between cofibrant objects is projectively cofibrant.

Corollary

For a pushout diagram $Y \xleftarrow{k} X \xrightarrow{l} Z$, its homotopy pushout is computed by finding $Y' \leftarrowtail X' \rightarrowtail Z'$ where X', Y', Z' are cofibrant objects, besides, two diagram are connected by weak equivalence, i.e., there is a diagram with verticle maps be weak equivalences.

$$\begin{tikzcd} Y' & X' \arrow[l, tail] \arrow[r, tail] & Z' \\ Y \arrow[u, "\wr"] & X \arrow[l, "k"] \arrow[r, "l"'] \arrow[u, "\wr"] & Z \arrow[u, "\wr"] \end{tikzcd}$$

or

$$\begin{tikzcd} Y' \arrow[d, "\wr"] & X' \arrow[l, tail] \arrow[r, tail] \arrow[d, "\wr"] & Z' \arrow[d, "\wr"] \\ Y & X \arrow[l, "k"] \arrow[r, "l"'] & Z \end{tikzcd}$$

In practice, here is a general way to find homotopy pushout:

1. Find a cofibrant object weak equivalence to the middle object $q : X' \to X$. (Direction is important. $X \xrightarrow{\sim} X'$ fails to have step 2)
2. Factor kq and lq by $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$, we obtain:

$$\begin{tikzcd} Y' \arrow[d, two heads, "\wr"] & X' \arrow[l, tail] \arrow[r, tail] \arrow[d, "\wr"] & Z' \arrow[d, two heads, "\wr"] \\ Y & X \arrow[l, "k"] \arrow[r, "l"'] & Z \end{tikzcd}$$

$\emptyset \rightarrowtail X' \rightarrowtail Y'$. So Y' **is a cofibrant** and so is Z'.

Compute the homotopy pushout of $* \leftarrow S^{n-1} \rightarrow *$. Recall model category of **Top**:

- $f : X \rightarrow Y$ is a cofibration if f is a retract of a map $\bar{f} : X \rightarrow Y'$ such that Y' is obtained from X by attaching cells.
- Initial object of **Top** is $\emptyset$
- All CW complexes are cofibrant objects

Remark

(All CW complexes are cofibrant objects) Let X be a CW complex. $\emptyset \rightarrow X$ is a retract of itself $\emptyset \rightarrow X$ and X is obtained from $\emptyset$ by attaching cells since X is a CW complex.

$$\begin{tikzcd} S^{n-1} \arrow[r, "id"] \arrow[d] & S^{n-1} \arrow[r, "id"] \arrow[d] & S^{n-1} \arrow[d] \\ D^n \arrow[r] \arrow[rr, bend right, "id"'] & * \arrow[r] & D^n \end{tikzcd}$$

This diagram shows $S^{n-1} \to D^n$ is a cofibration.

Clearly $D^n \to *$ is weak equivalence.

D^n and S^{n-1} is CW complex, so they are cofibrant objects.

There is a commutative diagram:

$$\begin{tikzcd}
D^n \arrow[d, "\wr"] & S^{n-1} \arrow[l, tail] \arrow[r, tail] \arrow[d, "id"] & D^n \arrow[d, "\wr"] \\
* & S^{n-1} \arrow[l] \arrow[r] & *
\end{tikzcd}$$

Hence, the homotopy pushout of $* \leftarrow S^{n-1} \rightarrow *$ is the pushout of $D^n \leftarrowtail S^{n-1} \rightarrowtail D^n$, which is S^n.

Thanks!