Energy bands and Higgs bundles

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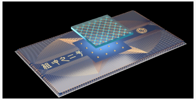
The research on superconductors is very popular.



Efficiency superconductors can be designed by controlling materials near singular points, where **energy bands** are gapless.

Studying **topology of energy bands** is significant because it is closely related to our daily life.

We can use **Higgs bundles** to describe the topology of the energy band. Higgs bundles are highly related to integrable systems, higher Teichmuller theory, geometric Langlands duality, etc. In this talk, I will discuss the picture of Higgs bundles and relations between Higgs bundles and Hopf bundles from view of eigenbundles.

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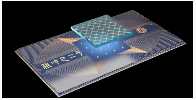  

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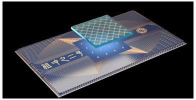

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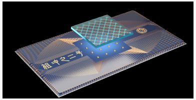

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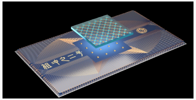

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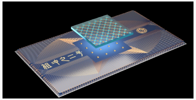

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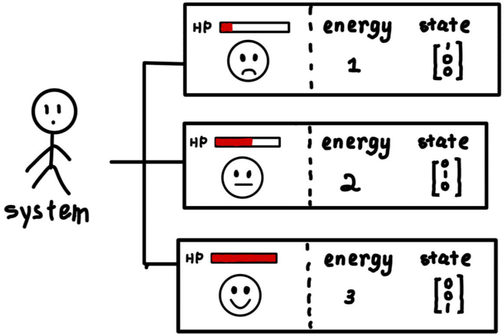
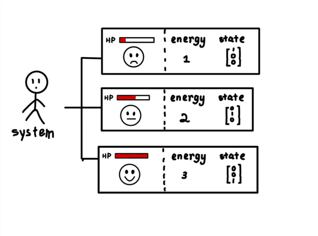
What are energy bands?

Most crucial physical information of a physical system: **energy** and **states**


system
HP
energy
state
1
HP
energy
state
2
HP
energy
state
3

This system is described by **Hamiltonian** $H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

- eigenvalues (possible energy): 1, 2, 3
- eigenvectors (possible states corresponding to energy): $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

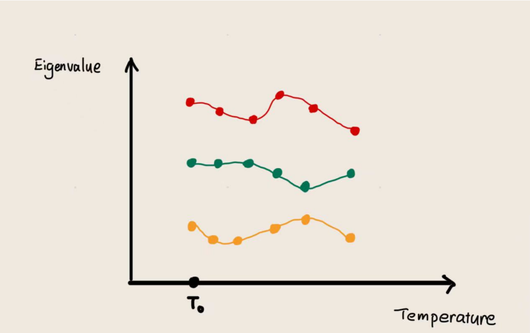


Tuning a system $\Leftrightarrow$ Parametrizing the Hamiltonian

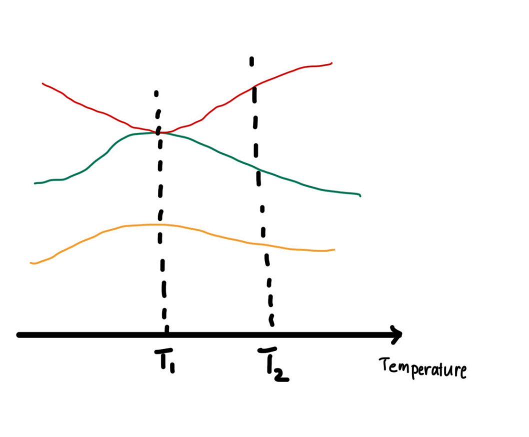
e.g., parametrized by temperature T:

$$H(T) = \begin{bmatrix} a_{11}(T) & a_{12}(T) & a_{13}(T) \\ a_{21}(T) & a_{22}(T) & a_{23}(T) \\ a_{31}(T) & a_{32}(T) & a_{33}(T) \end{bmatrix}$$

We can draw the **energy band**:



T_1 is a singular point.

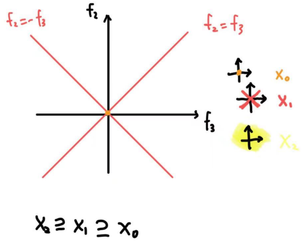


What is the topology of energy bands?

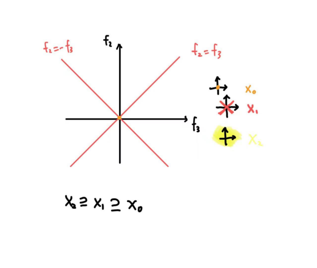
Summarize the content in one sentence: The topology of energy bands is the configuration of eigenvalues and eigenspaces.

Consider the matrix $H = \begin{bmatrix} f_3 & f_2 \\ -f_2 & -f_3 \end{bmatrix}$, $f_3,\ f_2 \in \mathbb{R}$.

Let's see the configuration of eigenvalues and eigenspaces.



$X_0 = \{(0,0)\}$, $X_1 = \{(f_3, f_3), (f_3, -f_3) | f_3 \in \mathbb{R}\}$, $X_2 = \mathbb{R}^2$



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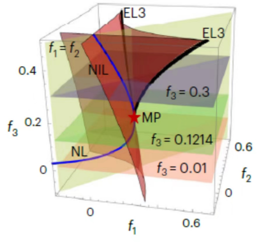
- X_0: $\lambda_+ = \lambda_- = \lambda$, $dim(E(\lambda)) = 2$
- $X_1 - X_0$: $\lambda_+ = \lambda_- = \lambda$, $dim(E(\lambda)) = 1$
- $X_2 - X_1$: $\lambda_+ \neq \lambda_-$, $dim(E(\lambda_\pm)) = 1$

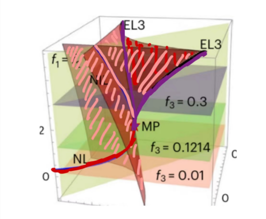
It's a stratified space with each stratum characterizing different behavior of eigenvalues and eigenvectors.

Consider configuration of eigenvalues & eigenspaces of

$$H = \begin{bmatrix} -f_1 - f_2 + 1 & -f_1 & -f_2 \\ f_1 & f_1 + f_3 & -f_3 \\ f_2 & -f_3 & f_2 + f_3 \end{bmatrix}$$

Singular points locally look like swallowtails.





Ref: Non-Hermitian swallowtail catastrophe revealing transitions among diverse topological singularities

Higgs bundles

Let S be a closed orientable surface of genus $g \geqslant 2$ and Σ be a Riemann surface structure on S.

Definition

A rank n **Higgs bundle** over Σ is a pair (E, ϕ) where E is a holomorphic vector bundle of rank n and $\phi \in H^0(\Sigma, End(E) \otimes K)$, called the **Higgs field**, where K is the cotangent bundle.

Equivalently, ϕ is a family of morphisms $\{\phi_x \in EndE_x \otimes K_x | x \in \Sigma\}$. ϕ_x **is a matrix of holomorphic one-form with eigenvalues valued in** K.

Ref: Qiongling Li. An introduction to Higgs bundles via harmonic maps. Symmetry, Integrability and Geometry: Methods and applications, May 2019

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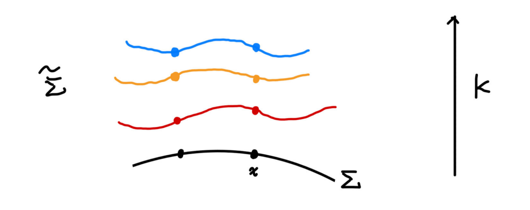
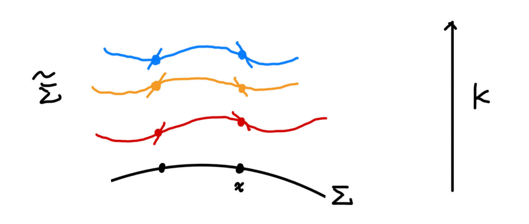
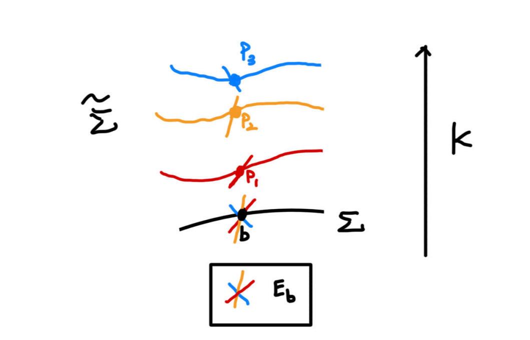
At each point $x \in \Sigma$, we can "draw" its eigenvalues, and finally, we can obtain **a graph of eigenvalues**, denoted by $\tilde{\Sigma}$. $\tilde{\Sigma}$ is a **branched cover over** Σ.

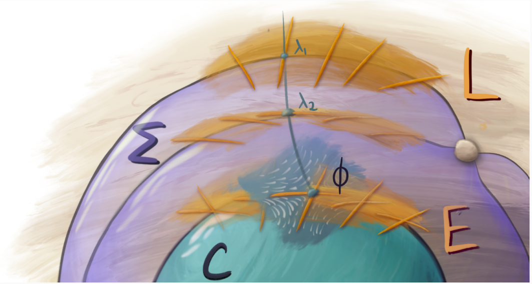


$L \to \tilde{\Sigma}$ is a line bundle: for $p \in \tilde{\Sigma}$, the fiber L_p is the eigenvector associated to eigenvalue p.



Recall that a linear transformation can split a vector space into a direct sum of eigenspaces. Hence, at a regular point b, $E_b = \oplus_i L_{p_i}$.





Ref:Hyperbolic band theory through Higgs bundles, Advanced in Mathematics, 409:108664, November 2022

Hopf bundles and Higgs bundles

We've known following Hopf bundles:

$S^0 \hookrightarrow S^1 \xrightarrow{2} S^1$

$S^1 \hookrightarrow S^3 \xrightarrow{\eta} S^2$

$S^3 \hookrightarrow S^7 \xrightarrow{\mu} S^4$

These Hopf bundles are nontrivial elements in $\pi_1(S^1)$, $\pi_3(S^2)$, $\pi_7(S^4)$.

They also arise from normalised eigenbundles of 2×2 matrices.

Ref: Principal bundles, Hopf bundles and eigenbundles

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$\downarrow$ generalize to

? 3-band pseudo-Hermitian Hamiltonian

Hopf bundle $S^0 \hookrightarrow S^1 \xrightarrow{2} S^1$ is a special Higgs bundle.

Classifying

Question: How to classify **Higgs bundles induced by parametrized Hamiltonians**?

Compute loops and find relations.

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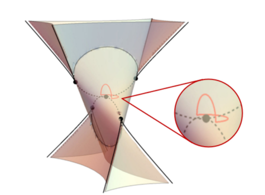
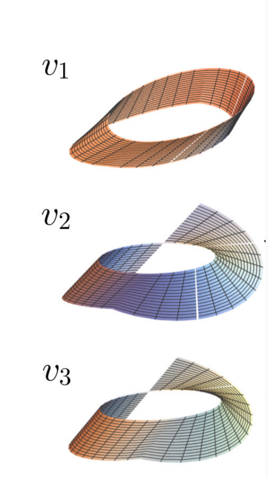
Consider singular points of the matrix $H_1 = \begin{bmatrix} 1 & f_1 & f_2 \\ -f_1 & -1 & f_3 \\ -f_2 & f_3 & -1 \end{bmatrix}$





Thanks!