Physical picture of tensor products and direct sums

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Abstract

Tensor products and direct sums of vector spaces are abstract concepts in linear algebra. In this article, we aim to elucidate these abstrct notions by developing into the physical manifestations of two systems: the 1/2-spin system and the two -particle system.

Contents

1 Introduction

In physics, we fequently encounter tensor products and direct sums in many cases of physics. Consider the two system: 1/2 spin system and two electron system. We know the state for one $|\uparrow\rangle$ and one $|\downarrow\rangle$ is $|\uparrow\rangle \otimes |\downarrow\rangle$, not $|\uparrow\rangle \oplus |\downarrow\rangle$, where $\otimes$ is tensor product. But in two-particle system, if the first particle has location $\vec{x_1}$ and the second particle has location x_2, the total state is $x_1 \oplus x_2$, not $x_1 \otimes x_2$, where $\oplus$ is the direct sum. They leave us frustrated: Why we employ tensor product in some cases and direct sums in other cases?

In Section 2, I will begin by listing some conditions that a 1/2-spin system must have. By drawing upon these physical insights, we will discoverthat tensor products in the realm of mathematics exactly agree with those fundmental conditions. This section will serve as a comprehensive introduction to tensor products, then you can find the tensor product here is so appropriate.

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In Section 3, I will outline key physical insights for a two-particle system. Through these exploration, the suitability for employing direct sums here will become clear. Here, I will introduce the direct sum in mathematics, highlighting their appropriateness in this context.

The selection between tensor products and direct sums is merely one illustrative example of the profound interplay between physics and mathematics. It serves as a stepping stone towards our ultimate objective in this article.

2 The state of two 1/2-spin system

Recall that the basis of the Hilbert space for a 1/2-spin system consists of two states: $|\uparrow\rangle$ and $|\downarrow\rangle$. In our study, we focus on two 1/2-spin system, which means we should understand how to describe the combined state of these two spins. That is, for example, how to describe the following state: One spin in the state of $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$ and the other in the state of $|\downarrow\rangle$? Since the combined state is relavent to the two states, we denote as:

$$\textbf{Combined state} = \textit{one state} \otimes \textit{the other state}$$

For example, $(\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle) \otimes |\downarrow\rangle$ means that the state that the first spin is $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$ and the second spin is $|\downarrow\rangle$.

Here, we use the symbol $\otimes$ to connect two states of spins. Up to this point, we have introduced this symbol $\otimes$ without developing its special meaning. It's merely a notation. In the following, I will show you what the symbol $\otimes$ is in mathematics.

To comprehend the meaning of $\otimes$, we need to explore its properties within physical picture. The following Example 2.1 and Example 2.2 serve as the illustrations of this exploration.

Example 2.1. Since $|\uparrow\rangle$ and $k|\uparrow\rangle$ $(0 \neq k \in \mathbb{C})$ represent the same stae, we consider $|\uparrow\rangle$ and $k|\uparrow\rangle$ as equivalent. Consequently, $\otimes$ satisfies: $(k|\uparrow\rangle) \otimes (|\downarrow\rangle) = |\uparrow\rangle \otimes (k|\downarrow\rangle) = k(|\uparrow\rangle \otimes |\downarrow\rangle)$

Example 2.2. Consider the state $(\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle) \otimes |\downarrow\rangle$. This state implies that we have a 50-50 posibility to find $\boxed{\text{the first spin in state } |\uparrow\rangle \text{ and the second spin in state } |\downarrow\rangle}$ and a 50-50 posibility to find $\boxed{\text{the first spin in state } |\downarrow\rangle \text{ and the second spin in state } |\downarrow\rangle}$.

Now, let's consider the state $\frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle) + \frac{1}{\sqrt{2}}(|\downarrow\rangle \otimes |\downarrow\rangle)$. This state also implies that we have a 50% chance of finding $\boxed{\text{the first spin in state } |\uparrow\rangle \text{ and the second spin in state } |\downarrow\rangle}$ and a 50% chance of finding $\boxed{\text{the first spin in state } |\downarrow\rangle \text{ and the second spin in state } |\downarrow\rangle}$. Because these two states have the same physical meaning, we conclude that $(\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle) \otimes |\downarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle) + \frac{1}{\sqrt{2}}(|\downarrow\rangle \otimes |\downarrow\rangle)$.

Example 2.1 and Example 2.2 suggest that $- \otimes -$ should be $\mathbb{C}$-bilinear.

Definition 2.3. Given V and W be two vector spaces over $\mathbb{C}$, a linear transformation $f : V \times W \to \mathbb{C}$ is $\mathbb{C}$-linear if f is $\mathbb{C}$-linear in each component.

With the property $- \otimes -$ should be $\mathbb{C}$-bilinear, we should try some common operations in mathematics. The most familiar operation for vector spaces we studied is the catesion product of vector spaces.

Definition 2.4. Given V and W be two $\mathbb{C}$-vector spaces. The cartesion product of V and W is a vector space $V \times W$ with the underlying set $\{(v, w)|v \in V, \quad w \in W\}$, the scalar multiplication $k(v, w) = (kv, kw), k \in \mathbb{C}$, and the addition $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$.

So the we do our first test:

Denote the two Hilbert spaces of 1/2-spin as V and W, respectively. Cartesian products $V \times W$ induces an operation $(-, -)$. Recall that we've defined $Combinedstate = onestate \otimes theotherstate$ and here we replace $- \otimes -$ as $(-, -)$. For example, the combined state $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle \otimes |\downarrow\rangle$is $(\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle, |\downarrow\rangle)$, where $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$ is a state in a Hilbert space and $|\downarrow\rangle$ in th other.

Suppose the cartisian product is $\mathbb{C}$-bilinear. Let $a \in V$ and $b \in W$, the combined state of two spins are (a, b). Then, we should have

$$(2a, b) = (a, 2b) = 2(a, b) \tag{1}$$
$$(a + a, 0 + b) = (a + 0, b + b) = (a, b) + (a, b) \tag{2}$$
$$\text{Eliminate } (a, b) \tag{3}$$
$$(a, 0) = (0, b) = (a, b) \tag{4}$$

$(-, -)$ is not $\mathbb{C}$-bilinear in mathematics. Besides, it's also not right in physics.The physical meaning of the equation $(a, 0) = (0, b) = (a, b)$ is the following three states are equal when we do observation:**(I)** one spin is vaccum and the other in the state of a**(II)**one spin is vaccum and the other in the state of b**(III)**one spin in state of v and the other in state of b. If a not equals to b, of course the above three states can not equal. So we obtain an obsurd result, meaning that $(-, -)$ is not the right choice for the symbol $\otimes$.

However, we can make a small change to the Cartesian product to satisfy the $\mathbb{C}$-bilinear property. The easiest way is to take quotient, as taking quotient allows for the classification of elements and provides a way for elements satisfy the required equations.

Definition 2.5. Let V be a vector space and U be a subspace of V. If $v - w \in U$, we say v and w in the same equivalence class, and denote this equivalence class as $\bar{v}$ or $v + M$. The quotient space V/M is a vector space with the underlying set $\{v + M|v \in V\}$, canonical scalar multiplication and addition.

Remark 2.6. If v and w are in the same equivalent class, $\bar{v} = \bar{w}$.

In vector space V, there is only one zero $0 \in V$. Roughly speaking, taking quotient V/M means we view M as zero.

Remark 2.7. To see that, you can coonsider the canonical map $\pi : V \to V/M$, $kern(\pi) = M$.

We need $(-,-)$ being $\mathbb{C}$-bilinear, so we want

• $(v_0 + v_1, w_0) - (v_0, w_0) - (v_1, w_0) = 0$
• $(v_0, w_0 + w_1') - (v_0, w_0) - (v_0, w_1) = 0$
• $(kv_0, w_0) - k(v_0, w_0) = 0$
• $(v_0, kw_0) - k(v_0, w_0) = 0$

Because taking quotient can view something as zero, we can throw the above elements to a set, generate a vector space on this set.we define

$$S \quad \text{be the subspace generated by} \begin{Bmatrix} (v_0 + v_1, w_0) - (v_0, w_0) - (v_1, w_0), \\ (v_0, w_0 + w_1) - (v_0, w_0) - (v_0, w_1), \\ (kv_0, w_0) - k(v_0, w_0), \\ (v_0, kw_0) - k(v_0, w_0) \end{Bmatrix}_{v_0, v_1 \in V, w_0, w_1 \in W, k \in \mathbb{C}}$$

Remark 2.8. Note that S is not the following set, but a vector space generated by the following set.

$$\begin{Bmatrix} (v_0 + v_1, w_0) - (v_0, w_0) - (v_1, w_0), \\ (v_0, w_0 + w_1) - (v_0, w_0) - (v_0, w_1), \\ (kv_0, w_0) - k(v_0, w_0), \\ (v_0, kw_0) - k(v_0, w_0) \end{Bmatrix}_{v_0, v_1 \in V, w_0, w_1 \in W, k \in \mathbb{C}}$$

Generated means you need to add more elements to let the new set becomes closed under scalar multiplication and addition.

Because we need to do quotient, S must be a vector space, not just a set.

Cartesian products $V \times W$ induces an operation $(-,-)$ on elements from V and W, which is not $\mathbb{C}$-bilinear. Now we do our second test. $\overline{(-,-)}$, the operation on elements from V and W induced $V \times W/S$.

Combined state = *one state* $\otimes$ *the other state*

, we replace $\otimes$ as $\overline{(-,-)}$ and check whether it's a good choice for the symbol $\otimes$.

Indeed, $\overline{(-,-)}$ is $\mathbb{C}$-bilinear. For example, by the construction of S, we have $(v_0 + v_1, w_0) - (v_0, w_0) - (v_1, w_0) \in S$, Hence $\overline{(v_0 + v_1, w_0)} = \overline{(v_0, w_0)} + \overline{(v_1, w_0)}$. The other conditions for $\mathbb{C}$-bilinear hold for the similar reason.

During our second test, $\overline{(-,-)}$ seems a good choice for the symbol $\otimes$. However, $\overline{(-,-)}$ still do not admit our command in another physical request.

Let's look carefully into the elements of $V \times W/S$. $V \times W/S = \{\overline{(v,w)} | v \in V, w \in W\}$. If $\otimes$ is $\overline{(-,-)}$, $V \otimes W = V \times W/S = \{v \otimes w | v \in V, w \in W\}$.

Clearly, $\{v \otimes w | v \in V, w \in W\}$ is not good representation for our real world. Consider two state in $V \otimes W$:

•state (I), denoted by $v \otimes w$:one spin in state v and another in state w
•state (II), denoted by $a \otimes b$: one spin in state a and another in state b

Clearly $V \times W/S = \{v \otimes w | v \in V, w \in W\}$ contains state (I) and (II). However, it doesnot contain the state $1/2v \otimes w + 1/2a \otimes b$, which means we have 50% to find one spin in state v, another in state w and 50% to find one spin in state a, another in state b. Hence, $\overline{(-,-)}$ is not a good choice for $\otimes$.

Our third test is only a few change to the second test. We denote the basis of V ($|\uparrow\rangle$ and $|\downarrow\rangle$) as v_1 and v_2. We denote the basis of W($|\uparrow\rangle$ and $|\downarrow\rangle$) as w_1 and w_2. Let F be the free module on $V \times W$, meaning every element in F can be written as the linear combination of elements in $V \times W$ with coefficients in $\mathbb{C}$,i.e., $F = \{\Sigma k_\lambda (v_\lambda, w_\lambda) | v_\lambda \in V, w_\lambda \in W\}$

This F consider all superpositions, so it's good. Then to make it $\mathbb{C}$-bilinear, we do quotient, whicih is same in our second test. Finally, we conclude that $V \otimes W = F/S$, where F/S is a vector space with a basis of $\{v \otimes w | v \in V, w \in W\}$.

The above construction is exactly tensor product in mathematics.

Definition 2.9. Let M and N be two R-modules. The tensor product of M and N is an R-module $M \otimes_R N$ together with an R-bilinear map $\tau : M \times N \to M \otimes_R N$ with the following properties: For any R-bilinear map $f : M \times N \to A$, there exists a unique map $\bar{f} : M \otimes_R N \to A$, such that $\bar{f}\tau = f$.

Theorem 2.10. Tensor products of any R-modules M and N exist.

The proof for Theorem 1 is to construct $M \otimes_R N$.The construction in Theorem 1 is: let F be the free R-module on the set $M \times N$. Let

$$S \quad \text{be the submodule generated by} \begin{Bmatrix} (v_0 + v_1, w_0) - (v_0, w_0) - (v_1, w_0), \\ (v_0, w_0 + w_1) - (v_0, w_0) - (v_0, w_1), \\ (kv_0, w_0) - k(v_0, w_0), \\ (v_0, kw_0) - k(v_0, w_0) \end{Bmatrix}_{v_0, v_1 \in V, w_0, w_1 \in W, k \in \mathbb{C}}$$

. Let $M \otimes_R N = F/S$ and $\tau : M \times N \to M \otimes N$ defined by $\tau(m, n) = m \otimes n$, where $m \otimes n$ is the equivalent class of (m, n) in F/S. We do not tell details for the work to check univeral property. It's exactly what we've done just from physical picture. That is the symbol $\otimes$ we use is exactly tensor product.

Remark 2.11. From this construction, we find that $M \otimes_R N$ is generated by $m \otimes n$, which we call simple tensors.

Here I provide some statements about tensor product without proof, and tell the physical meaning for those statements.

Theorem 2.12. The tensor product is unique up to unique isomorphism.

It's easy to understand, because the combined Hilbert space is the unique we observe. It's hard to understand the state space for aligning two 1/2 spins is not unique. That may lead to this absurd result: Amely may observe different phenomenon as Bob.

Property 2.13. Let M_1, $M2$ be R-modules. $M_1 \otimes_R M_2 \simeq M_2 \otimes_R M_1$

Particales are identical, we can tell wich one has the Hilbert space M and which one has the Hilbert space N. We only know they have Hilbert space M and N, respectively. So we can not disdinguish the two spaces $M \otimes_R N$ and $N \otimes_R M$.

Property 2.14. Let A, B, and C be R-module homomorphisms. $(A \otimes_R B) \otimes_R C \simeq A \otimes_R (B \otimes_R C)$.

It's still natural because we can not disdinguish particles.

Property 2.15. Let $f : A \to B$ and $g : C \to D$ be two maps of $\mathbb{C}$-modules. Then there exists a map $f \otimes g : A \otimes C \to B \otimes D$ such that $(f \otimes g = (f(a) \otimes g(b)))$.

$f \otimes g$ describes the transformation on each spin.

You may familiar with the tensor product of two matrics. You can see more detail in Appendix.

3 Two-particle system

Consider two-particle system. For simplicity, we only consider 3d space dimension. Let V be the state space of one particle, that means, for any $(x, y, z) \in V$, the particle locates at (x, y, z).

What's the total vector space of two particle system if the space of each electron is V and W, respectively? **Combined state** = *one state*$\oplus$*the other state*Let's denote the total space as $V \oplus W$, here $\oplus$ is just a notation.

We'll find what $\oplus$ is through physical insight.

Example 3.1. Suppose one particle locates at $(1, 1, 1)$ and the other locates at $(10, 10, 10)$. The combined state is $(1, 1, 1) \oplus (10, 10, 10)$. The state $(2(1, 1, 1)) \oplus (10, 10, 10)$ means one particle locates at $(2, 2, 2)$ and the other at $(10, 10, 10)$. The state $(1, 1, 1) \oplus (2(10, 10, 10))$ means one particle locates at $(1, 1, 1)$ and the other locates at $(20, 20, 20)$. Hence $(2(1, 1, 1)) \oplus (10, 10, 10)$ is not equal to $(1, 1, 1) \oplus (2(10, 10, 10))$. Similarly, $rv \oplus w \neq v \oplus rw$, for any $r \in \mathbb{C}$, $w \in W, v \in V$.

$- \oplus -$ is not $\mathbb{C}$-bilinear. So the symbol $\oplus$ can not be the tensor product in Section 2.

We are in the classical mechanics, so everything is determined. There is no propability, or superposition any more.The system should have this property: Let $\{x_1, x_2\}$be the state that the first particle locates at x_1 and the second particle locates at x_2.

Definition 3.2. Let V and W be the subspaces of the vector space L. We say $V + W$ is the direct sum, denoted by $V \oplus W$, if for any $x \in V \oplus W$, there exists unique v and w, such that $x = v + w$.

Look the following definition, you may find they are telling one thing: The combined system can be taken into two parts and any part and can be represented uniquely. So the direct sum is the right choice for $\oplus$.

4 Conclusion

The total state of two spins align is using tensor product. The total state of two electrons align is using direct sum.

5 Appendix:Tensor products of matrics

$f[v_1 v_2] = [v_1 v_2]\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, thus $f(v_1) = a_{11}v_1 + a_{21}v_2$ and $f(v_2) = a_{12}v_1 + a_{22}v_2$.

$g[w_1 w_2] = [w_1 w_2]\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, thus $g(w_1) = b_{11}w_1 + b_{21}w_2$ and $g(v_2) = b_{12}w_1 + b_{22}w_2$.

$\otimes$ is $\mathbb{C}$-bilinear. Hence,

$$\begin{aligned} f \otimes g(v_1 \otimes w_1) &= f(v_1) \otimes g(w_1) \\ &= (a_{11}v_1 + a_{21}v_2) \otimes (b_{11}w_1 + b_{21}w_2) \\ &= a_{11}b_{11}v_1 \otimes w_1 + a_{21}b_{11}v_2 \otimes w_1 \\ &\quad + a_{11}b_{21}v_1 \otimes w_2 + a_{21}b_{21}v_2 \otimes w_2 \end{aligned}$$

$$\begin{aligned} f \otimes g(v_2 \otimes w_1) &= f(v_2) \otimes g(w_1) \\ &= (a_{12}v_1 + a_{22}v_2) \otimes (b_{11}w_1 + b_{12}w_2) \\ &= a_{12}b_{11}v_1 \otimes w_1 + a_{22}b_{11}v_2 \otimes w_1 \\ &\quad + a_{12}b_{21}v_1 \otimes w_2 + a_{22}b_{12}v_2 \otimes w_2 \end{aligned}$$

$$\begin{aligned} f \otimes g(v_1 \otimes w_2) &= f(v_1) \otimes g(w_2) \\ &= (a_{11}v_1 + a_{21}v_2) \otimes (b_{12}w_1 + b_{22}w_2) \\ &= a_{11}b_{12}v_1 \otimes w_2 + a_{21}b_{12}v_2 \otimes w_1 \\ &\quad + a_{11}b_{22}v_1 \otimes w_2 + a_{21}b_{22}v_2 \otimes w_2 \end{aligned}$$

$$\begin{aligned} f \otimes g(v_2 \otimes w_2) &= f(v_2) \otimes g(w_2) \\ &= (a_{12}v_1 + a_{22}v_2) \otimes (b_{12}w_1 + b_{22}w_2) \\ &= a_{12}b_{12}v_1 \otimes w_2 + a_{22}b_{12}v_2 \otimes w_1 \\ &\quad + a_{12}b_{22}v_1 \otimes w_2 + a_{22}b_{22}v_2 \otimes w_2 \end{aligned}$$

$$
\begin{aligned}
f \otimes g[v_1 \otimes w_1 \quad v_2 \otimes w_1 \quad v_1 \otimes w_2 \quad v_2 \otimes w_2] \\
= g[v_1 \otimes w_1 \quad v_2 \otimes w_1 \quad v_1 \otimes w_2 \quad v_2 \otimes w_2] \\
\begin{bmatrix} a_{11}b_{11} & a_{12}b_{11} & a_{11}b_{12} & a_{12}b_{12} \\ a_{21}b_{11} & a_{22}b_{11} & a_{21}b_{12} & a_{22}b_{12} \\ a_{11}b_{21} & a_{12}b_{12} & a_{11}b_{22} & a_{12}b_{22} \\ a_{21}b_{21} & a_{22}b_{12} & a_{21}b_{22} & a_{22}b_{22} \end{bmatrix}
\end{aligned}
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