Contents

John Jones' construction of manifold in dimension 30 with Kervaire invariant one

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Notation:

- $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$
- Σ_t: the group of permutations of a set with t elements
- Denote $\Sigma = \{\sigma, 1\}$, where σ is the nontrivial element.
- Let τ_Y be the tangent bundle of a manifold Y
- Let $L = \{(l, z) \mid l \in \mathbb{RP}^\infty, z \in l\}$. Let $H : L \to \mathbb{RP}^\infty$ denote the tautological/canonical/Hopf line bundle over $\mathbb{RP}^\infty$. The restriction over $\mathbb{RP}^n$ is denoted by H_n
- $tot(\cdot)$ means the total space of a bundle
- $Prin(X, G)$ denotes the set of principal G-bundles over X
- The important constructions are colored in blue

1 Kervaire Invariant

1.1 Arf Invariant

This section is a summary of [5].

Naively, a quadratic form is a linear k-function over k-vector space $q : V \to k$ for a given field k such that $q(tx) = t^2 q(x)$ for any $t \in k$ and $x \in V$.

Problem 1.1. When $k = \mathbb{Z}_2$, the condition should be $q(0x) = 0$ and $q(1x) = q(x)$ for all $x \in V$, which is just $q(0) = 0$. This condition appears too weak, and we must seek a new definition for the quadratic form.

This problem leads to the definition of a quadratic form on a $\mathbb{Z}_2$-vector space.

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Definition 1.2. Let V be a $\mathbb{Z}_2$-vector space. A function $q : V \to \mathbb{Z}_2$ is said to be a quadratic form if $I : V \times V \to \mathbb{Z}_2$ defined by $I(x, y) = q(x+y) - q(x) - q(y)$ is a bilinear map. In this case, we call q a quadratic refinement of the bilinear form I, and I is the associated bilinear form of q. We say this quadratic form is nondegenerate if the associated bilinear form I is nondegenerate, i.e., $V \to Hom(V, \mathbb{Z}_2)$, $v \mapsto I(v, -)$ is an isomorphism.

Remark 1.3. The quadratic refinement of the bilinear form is not unique. An example is in **Construction 1.9**. □

Definition 1.4. Let V be a $\mathbb{Z}_2$-vector space and I be a bilinear form on V with quadratic refinement $q : V \to \mathbb{Z}_2$. If V has a basis $\{a_i, b_i | i = 1, 2, \cdots s\}$ fulfilling $I(a_i, a_j) = I(b_i, b_j) = 0$ and $I(a_i, b_j) = I(a_j, b_i) = \delta_{ij}$, then we call this basis a symplectic basis for the bilinear form I.

Fact 1.5. If a $\mathbb{Z}_2$-vector space admits a nondegenerate quadratic form $q : V \to \mathbb{Z}_2$, then a symplectic basis exists. In particular, $\dim V$ is even.

Definition 1.6. Let $q : V \to \mathbb{Z}_2$ be a nondegenerate quadratic form on $\mathbb{Z}_2$-vector space V. By **Fact 1.5**, there exists a symplectic basis $\{a_i, b_i | i = 1, 2, \cdots, s\}$ for the associated bilinear form I. The Arf-invariant of q is defined as

$$\mathrm{Arf}(q) = \sum_{i=1}^{s} q(a_i) q(b_i).$$

For the remaining part, let us focus on what the Arf invariant characterizes.

Definition 1.7. Let $q, q' : V \to \mathbb{Z}_2$ be two quadratic forms. We say that q is *equivalent* to q', denoted by $q \sim q'$, if there exists an automorphism $f : V \to V$ such that $q \circ f = q'$.

Clearly, this defines an equivalence relation. The following fact shows that the Arf invariant is invariant under this equivalence relation.

Fact 1.8. The Arf invariant is a function on the equivalence classes of quadratic forms.

Construction 1.9. (For a bilinear map, its quadratic refinement is not unique.)

Let $W = \mathbb{Z}_2 a_1 \oplus \mathbb{Z}_2 b_1$ be a $\mathbb{Z}_2$-vector space with basis $\{a_1, b_1\}$. A bilinear map on $V \times V$ is determined by

$$I(a_1, a_1), \quad I(a_1, b_1), \quad I(b_1, a_1), \quad I(b_1, b_1).$$

A bilinear form admitting a quadratic refinement should satisfy

$$I(a_1, a_1) = I(b_1, b_1) = 0, \qquad I(a_1, b_1) = I(b_1, a_1).$$

If we further require that this bilinear form be nondegenerate, then the only possibility is

$$I(a_1, a_1) = I(b_1, b_1) = 0, \qquad I(a_1, b_1) = I(b_1, a_1) = 1.$$

As a set, $W = \{a_1, b_1, a_1 + b_1, 0\}$. Therefore, a quadratic refinement of q is determined by the values of $q(a_1)$, $q(b_1)$, and $q(a_1 + b_1)$.

Define two quadratic forms as follows:

$$q_0(a_1) = q_0(b_1) = 0, q_0(a_1 + b_1) = 1$$

and

$$q_1(a_1) = q_1(b_1) = q_1(a_1 + b_1) = 1$$

We can easily check that both are quadratic refinements of I. Moreover, q_0 is not equivalent to q_1, since q_0 sends most elements to 0 while q_1 sends most elements to 1. An automorphism on a vector space does not alter the proportions of the values. Therefore, equivalent quadratic forms map elements to 0 and 1 in the same ratio. □

Construction 1.10. Let U be a $\mathbb{Z}_2$-vector space that admits a nondegenerate quadratic form, and $\dim U = 2m$, $m \in \mathbb{N}$. By **Fact 1.5**, U has a symplectic basis

$$\{a_i, b_i \mid i = 1, 2, \ldots, m\}$$

for the associated bilinear form I.

Then

$$U = (\mathbb{Z}_2 a_1 \oplus \mathbb{Z}_2 b_1) \oplus (\mathbb{Z}_2 a_2 \oplus \mathbb{Z}_2 b_2) \oplus \cdots \oplus (\mathbb{Z}_2 a_m \oplus \mathbb{Z}_2 b_m) = W_1 \oplus W_2 \oplus \cdots \oplus W_m,$$

where $W_i = \mathbb{Z}_2 a_i \oplus \mathbb{Z}_2 b_i$. Each W_i is a 2-dimensional vector space, so W_i has two quadratic forms q_0, q_1 in **Construction 1.9**.

We have quadratic forms:

$$mq_0 := q_0 \oplus q_0 \oplus \cdots \oplus q_0 : U = W_1 \oplus W_2 \oplus \cdots \oplus W_m \to \mathbb{Z}_2,$$

and

$$q_1 + (m-1)q_0 := q_1 \oplus q_0 \oplus \cdots \oplus q_0 : W_1 \oplus \cdots \oplus W_m \to \mathbb{Z}_2.$$

These two quadratic forms are the standard forms for nondegenerate quadratic forms. □

Fact 1.11. (Classification by the Arf invariant) The Arf invariant classifies all nondegenerate quadratic forms. More explicitly,

$$\mathrm{Arf}(q) = 0 \iff q \text{ maps the majority of elements to } 0 \iff q \sim mq_0,$$
$$\mathrm{Arf}(q) = 1 \iff q \text{ maps the majority of elements to } 1 \iff q \sim q_1 + (m-1)q_0.$$

1.2 Kervaire Invariant

In this section, $H^i(M) := H^i(M, \mathbb{Z})$.

Let (M, F) be a framed manifold. Using framing, we can define a quadratic form $\phi : H^n(M; \mathbb{Z}_2) \to \mathbb{Z}_2$. Let us only define the Kervaire invariant for a special case. The reference here is [4].

Let M^{10} be a closed triangulable 4-connected manifold. Let $\Omega := \Omega S^6$ be the loop space of S^6.

Fact 1.12. $H^5(\Omega S^6) = \mathbb{Z}e_1$, $H^{10}(\Omega S^6) = \mathbb{Z}e_2$. Let $\pi : \Omega \times \Omega \to \Omega$ be the product of loops, then $\pi^*(e_1) = e_1 \otimes 1 + 1 \otimes e_1$ and $\pi^*(e_2) = e_2 \otimes 1 + 1 \otimes e_2 + e_1 \otimes e_1$.

Fact 1.13. Let X be any element in $H^5(M)$. There exists a map $f : M \to \Omega$ such that $f^*(e_1) = X$.

Construction 1.14. Let $X \in H^5(M)$, then there exists a map $f_X : M \to \Omega$ such that ${f_X}^*(e_1) = X$. Let $u_2 \in H^{10}(\Omega; \mathbb{Z}_2)$ be the reduction modulo 2 of $e_2 \in H^{10}(\Omega)$ and $[M]$ be the generator of $H_{10}(M; \mathbb{Z})$. Then we define a map $\phi_0 : H^5(M) \to \mathbb{Z}_2$ by $\phi_0(X) = f_X^*(u_2)[M]$. □

Remark 1.15. $\phi_0(X)$ does not depend on the choice of $f_X : M \to \Omega$. □

Fact 1.16. $\phi_0(X + Y) = \phi_0(X) + \phi_0(Y) + X \cdot Y$ for $X, Y \in H^5(M)$. Then ϕ_0 induces a map $\phi : H^5(M; \mathbb{Z}_2) \to \mathbb{Z}_2$ also satisfying $\phi(x+y) = \phi(x) + \phi(y) + x \cdot y$ for $x, y \in H^5(M; \mathbb{Z}_2)$.

Since $I(x, y) = \phi(x + y) - \phi(x) - \phi(y) = x \cdot y$ is bilinear, so ϕ is a quadratic form.

Definition 1.17. Define the Kervaire invariant by $K(M, F) := \mathrm{Arf}(\phi)$.

Remark 1.18. $K(M, F) = 1$ if and only if ϕ sends the majority of elements of $H^5(M)$ to $1 \in \mathbb{Z}_2$. □

2 Construction in Dimension 30

2.1 Construction of extended power

If a framed manifold M has Kervaire invariant one, then $\dim M = 2, 6, 14, 30, 62, 126$.

Example 2.1. $S^1 \times S^1$, $S^3 \times S^3$, $S^7 \times S^7$ can be framed to have Kervaire invariant one.

This section introduces a construction in dimension 30 by [3]. The construction is of the form known as an extended power.

Definition 2.2. Let Y be a manifold with $G \subset \Sigma_t$ acting on it freely. Let N be a topological space and G acts on N^t by permutation. Then we can construct

$$Y_G(N) := Y \times_G N^t,$$

which is called the extended power.

Remark 2.3. Here is an advantage of considering extended power. That is, the dimension of extended power is computable:

$$\dim(Y \times_G N^t) = \dim Y + t \dim N.$$

G acts on Y and N^t freely. So G acts freely on $Y \times N^t$. Review Proposition 1.40 in [2], condition $(\star)$ automatically holds for the finite group G. Hence,

$Y \times N^t \to Y \times_G N^t$ is a normal covering space. By the property of covering space,

$$\dim(Y \times_G N^t) = \dim(Y \times N^t) = \dim Y + t \dim N.$$

□

Fact 2.4. Let G and H be any two Lie groups. $EH \times_H BG$ is a model for $B(G \rtimes H)$.

Construction 2.5. Let $\overline{X} := \mathbb{RP}^2 \# T^2$. Let $G = \Sigma_2 \wr \Sigma_2 = (\Sigma_2 \times \Sigma_2) \rtimes \Sigma_2 \subset \Sigma_4$.

Let $a : \mathbb{RP}^2 \subset B\Sigma_2 \xrightarrow{\iota} B\Sigma_2 \times B\Sigma_2 \xrightarrow{j} E\Sigma_2 \times_{\Sigma_2} (B\Sigma_2 \times B\Sigma_2) = BG$, where j is inclusion. The last equation holds by **Fact 2.4**.

Let $b : T^2 = S^1 \times S^1 \hookrightarrow B\Sigma_2 \times B\Sigma_2 \xrightarrow{1 \times_{\Sigma_2} \Delta} E\Sigma_2 \times_{\Sigma_2} (B\Sigma_2 \times B\Sigma_2) =: BG$. Here $1 \times_{\Sigma_2} \Delta$ is constructed as following:

- Define $E\Sigma_2 \times B\Sigma_2 \xrightarrow{1 \times \Delta} E\Sigma_2 \times (B\Sigma_2 \times B\Sigma_2)$. Both sides have a Σ_2-action as follows. For $(a, b) \in E\Sigma_2 \times B\Sigma_2$, $(c, (d, e)) \in E\Sigma_2 \times (B\Sigma_2 \times B\Sigma_2)$, we have $\sigma \cdot (a, b) = (\sigma \cdot a, b)$ and $\sigma(c, (d, e)) = (\sigma \cdot c, (e, d))$.
- The following diagram shows $1 \times \Delta$ is Σ_2-equivalence:

$$\begin{array}{ccc}
(a,b) & \longmapsto & (a,(b,b)) \\
\downarrow & & \downarrow \\
(\sigma \cdot a, b) & \longmapsto & (\sigma \cdot a, (b,b))
\end{array}$$

$$\begin{array}{ccc}
E\Sigma_2 \times B\Sigma_2 & \longrightarrow & E\Sigma_2 \times (B\Sigma_2 \times B\Sigma_2) \\
\sigma\cdot \downarrow & & \downarrow \sigma\cdot \\
E\Sigma_2 \times B\Sigma_2 & \longrightarrow & E\Sigma_2 \times (B\Sigma_2 \times B\Sigma_2)
\end{array}$$

So $1 \times \Delta$ is Σ_2-equivalence. Hence, we can obtain a map

$$1 \times_{\Sigma_2} \Delta : (E\Sigma_2 \times B\Sigma_2)/\Sigma_2 \to (E\Sigma_2 \times (B\Sigma_2 \times B\Sigma_2))/\Sigma_2$$

which is

$$1 \times_{\Sigma_2} \Delta : B\Sigma_2 \times B\Sigma_2 \to E\Sigma_2 \times_{\Sigma_2} (B\Sigma_2 \times B\Sigma_2).$$

Then we can construct $c : \overline{X} = \mathbb{RP}^2 \# T^2 \xrightarrow{\text{contraction}} \mathbb{RP}^2 \vee T^2 \xrightarrow{a \vee b} BG$. Let $X \to \overline{X}$ be the principal G-bundle classified by c. □

We consider the extended power $X_G(S^7)$. Note that

$$\dim X_G(S^7) = \dim X + 4 \dim S^7 = 2 + 4 \times 7 = 30$$

by **Remark2.10**.

Remark 2.6. Here are some reasons for choosing such an $\overline{X}$. One reason is that $H^*(\overline{X})$ is easily computed. Besides, the goal of finding $\overline{X} \to BG$ can be divided into constructing $\mathbb{RP}^2 \to BG$ and $T^2 \to BG$. □

2.2 Show $X_G(S^7)$ is framed

Let $\xi : X_G(\mathbb{R}) \to \overline{X}$ be the 4-dimensional bundle. Then by **Theorem A** of [3], to show that $X_G(S^7)$ is framed is equivalent to showing that $\tau\overline{X} + 7\xi$ is stably trivial. Since all stable bundles over a surface are classified by their Stiefel–Whitney classes, it suffices to show

$$w(\tau\overline{X} + 7\xi) = w(\tau\overline{X})w(\xi)^7 = 1.$$

$H^1(\overline{X}) = H^1(\mathbb{RP}^2) + H^1(T^2) = \mathbb{Z}_2 u + \mathbb{Z}_2 x_1 + \mathbb{Z}_2 x_2$, where u is the generator coming from $H^1(\mathbb{RP}^2)$, and x_1, x_2 are the generators coming from $H^1(T^2)$. $H^2(\overline{X}) = \mathbb{Z}_2 u^2$. Note that $u^2 = x_1x_2 \neq 0$, $x_1^2 = x_2^2 = ux_1 = ux_2 = u^3 = 0$.

By [1] $w(\tau\overline{X}) = w(\tau\mathbb{RP}^2 \# T^2) = w(\tau\mathbb{RP}^2) + w(\tau T^2) - 1 = 1 + u + u^2 + 1 - 1 = 1 + u + u^2$.

Let $r : G \to O(4)$ be the permutation representation. Let ρ be the bundle classified by Br.

Remark 2.7. Clearly, a normal bundle of T^2 is trivial. So $w(\nu T^2) = 1$. By $w(\tau T^2 \oplus \nu T^2) = w(\tau T^2)w(\nu T^2) = 1$, we get $w(\tau T^2) = 1$. The computation of $w(\tau\mathbb{RP}^2)$ is more complicated. Note that $\tau\mathbb{RP}^2 \oplus \varepsilon^1 = \oplus^3 H_2$ where H_2 is the tautological bundle over $\mathbb{RP}^2$. So $w(\tau\mathbb{RP}^2) = w(H_2)^3 = (1+u)^3 = 1 + u + u^2$. For more details, see [5]. □

Claim 2.8. $\xi = c^*\rho$.

ρ is the bundle classified by Br, so by **Remark 2.10**, $\rho : EG \times_{G,r} O(4) \to BG$. Then $c^*\rho = X \times_{G,r} O(4) \to X$ by **Remark 2.11**, which is a principal $O(4)$-bundle. There exists a corresponding $O(4)$-bundle with fiber $\mathbb{R}^4$, which has total space $(X \times_{G,r} O(4)) \times_{O(4)} \mathbb{R}^4 = X \times_{G,r} (O(4) \times_{O(4)} \mathbb{R}^4) = X \times_{G,r} \mathbb{R}^4 = X_G(\mathbb{R})$. Hence $c^*\rho : X_G(\mathbb{R}) \to X$ is ξ.

Remark 2.9. Let $p \in P$, $g \in G$, $h \in H$. For $\theta : H \to G$, we can define the balanced product $P \times_{H,\theta} G := P \times G/\sim$, where $(ph, g) \sim (p, hg)$. Equivalently, $P \times_{H,\theta} G = (P \times G)/H$, where $h(p,g) = (ph^{-1}, hg)$. Balanced product satisfies associativity and $W \times_G G \simeq G$ for any group G and G-space W. □

Remark 2.10. Let us consider a more general question: Let H, G be any groups, X be any topological space. Given a group homomorphism $\theta : H \to G$, what's $B\theta$? Here provides a construction of the map $B\theta$ (and this construction works).

Goal: construct an element in $\mathrm{Hom}(BH, BG)$. By Yoneda lemma, we need to construct an element in $\mathrm{Hom}([-, BH], [-, BG])$, i.e., we need to construct $\phi_X : [X, BH] \to [X, BG]$ for any X. By isomorphisms $\mathrm{Prin}(X, H) \simeq [X, BH]$ and $\mathrm{Prin}(X, G) \simeq [X, BG]$, we need to construct the map induced by ϕ, $\bar{\phi}_X : \mathrm{Prin}(X, H) \to \mathrm{Prin}(X, G)$. Given a principal H-bundle $P \to X$, we want to construct a principal G-bundle, using data $P \to X$ and $\theta : H \to G$. Note that $\theta : H \to G$ makes H acting left on G. So we can form an H-bundle with fiber G: $P \times_{H,\theta} G \to X$. Clearly we can view $[P \times_{H,\theta} G \to X]$ as a G-bundle with G

action on fiber G by multiplication. So $[P \times_{H,\theta} G \to X] \in B(X, G, G, m_G) =: \text{Prin}(X, G)$.

By Yoneda lemma, $\text{Hom}([-, BH], [-, BG]) \to \text{Hom}(BH, BG)$ defined by $\Psi \mapsto \Psi_{BH}(\text{id})$ is an isomorphism. Let $\Psi = \phi$ and define $B\theta := \phi_{BH}(\text{id})$, the image of ϕ under Yoneda map. Since $\bar{\phi}_X$ is induced by ϕ_X, we have commutative diagram:

$$\begin{array}{ccc} [BH, BH] & \xrightarrow{\phi_{BH}} & [BH, BG] \\ \downarrow \simeq & & \downarrow \simeq \\ Prin(BH, H) & \xrightarrow[\bar{\phi}_{BH}]{} & Prin(BH, G) \end{array}$$

$$\begin{array}{ccc} id & \longmapsto & \phi_{BH}(id) =: B\theta \\ \downarrow & & \downarrow \\ id^* EH = [EH \to BH] & \longmapsto & [EH \times_{H,\theta} G \to BH] \end{array}$$

From this diagram, even we do not know what $B\theta$ is, we can know $B\theta$ classifies principal G-bundle $EH \times_{H,\theta} G \to BH$. □

Remark 2.11. Consider this problem: Let G be any group, X be any topological space, and $\pi : E \to E/G$ be a bundle. The pullback of π along $c : X \to E/G$ is M. Let L be the pullback of $E \times_G F \to E/G$ along c. Then what is the relationship between L and M?

By construction,

$$\begin{aligned} L &= X \times_{E/G} (E \times_G F) \\ &= \{(x, e, f) \mid x \in X, e \in E, f \in F, c(x) = [e]_G\}/(x, e, f) \sim (x, eg, g^{-1} f), \end{aligned}$$

for any $g \in G$.

$$\begin{aligned} M \times_G F &= (X \times_{E/G} E) \times_G F \\ &= \{(x, e, f) \mid x \in X, e \in E, f \in F, c(x) = [e]_G\}/(x, e, f) \sim (x, eg, g^{-1} f) \end{aligned}$$

for any $g \in G$.

So $L = M \times_G F$. □

Since $\xi = c^*\rho$, we need to compute $w(c^*\rho)$. To compute $w(c^*\rho)$, we need to compute $a^*\rho$ and $b^*\rho$.

$G = \Sigma_2 \wr \Sigma_2 \subset \Sigma_4$ is the subgroup generated by $(12), (34), (13)(24)$ by **Remark 2.12**. Let $i_1 : \Sigma_2 \times \Sigma_2 \to G$ be the inclusion of the subgroup generated by (12) and (34). Let $i_2 : \Sigma_2 \times \Sigma_2 \to G$ be the inclusion of the subgroup generated by (12)(34) and (13)(24). Let $s : \Sigma_2 \to O(2)$ be a permutation representation.

Remark 2.12. $G = (\Sigma_2 \times \Sigma_2) \rtimes \Sigma_2$, where the multiplication is

$$((h_0, h_1), x) \cdot ((l_0, l_1), y) = ((y \cdot (h_0, h_1))(l_0, l_1), xy)$$

and

$$\begin{cases} \sigma(h_0, h_1) & = (h_1, h_0), \\ 1(h_0, h_1) & = (h_0, h_1) \end{cases}$$

G is isomorphic to the subgroup in Σ_4 generated by $(12), (34), (13)(24)$. The correspondence is as follows:

$((1,1),1)$	1
$((\sigma,1),1)$	(12)
$((1,\sigma),1)$	(34)
$((\sigma,\sigma),1)$	$(12)(34)$
$((1,1),\sigma)$	$(13)(24)$
$((\sigma,1),\sigma)$	(1423)
$((1,\sigma),\sigma)$	(1324)
$((\sigma,\sigma),\sigma)$	$(14)(23)$

Table 1: Correspondance

□

Claim 2.13. Bs classified the bundle $H + \varepsilon^1$.

By **Remark2.10**, Bs induces map classifies the principal $O(2)$-bundle $E\Sigma_2 \times_{\Sigma_2} O(2) \to B\Sigma_2$. This principal $O(2)$-bundle corresponds to an element in $B(B\Sigma_2, O(2), \mathbb{R}^2, \rho)$, where $\rho : O(2) \times \mathbb{R}^2 \to \mathbb{R}^2$, $(M, x) \mapsto M \cdot x$. The associated $O(2)$-bundle with fiber $\mathbb{R}^2$ is $E\Sigma_2 \times_{\Sigma_2} O(2) \times_{O(2)} \mathbb{R}^2 = E\Sigma_2 \times_{\Sigma_2} \mathbb{R}^2 \to B\Sigma_2$.

Write $u = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}] \in \mathbb{R}^2$ and $v = [\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}] \in \mathbb{R}^2$. We observe that $\sigma u = u$ and $\sigma v = -v$. (Here u and v are more symmetric than a standard basis.)

Check that the total spaces are the same:

- $E\Sigma_2 \times_{\Sigma_2} \mathbb{R}^2 = \{(z, xu + yv) \mid z \in E\Sigma_2 = S^\infty, x, y \in \mathbb{R}\}/\sim$, where $(z, xu + yv) \sim (z\sigma^{-1}, \sigma(xu + yv)) = (-z, xu - yv)$. Let $w = z \cdot y$ and $x = t$, then $E\Sigma_2 \times_{\Sigma_2} \mathbb{R}^2 = \{(w, t) \mid w \in \mathbb{R}^\infty, t \in \mathbb{R}\}$.
- $\text{tot}(H + \varepsilon^1) = \{((l, w), t) \mid l \in \mathbb{RP}^\infty, w \in l \subset \mathbb{R}^\infty, t \in \mathbb{R}\} = \{(w, t) \mid w \in \mathbb{R}^\infty, t \in \mathbb{R}\} = E\Sigma_2 \times_{\Sigma_2} \mathbb{R}^2$.

Check that the bundle maps are the same:

- $H + \varepsilon^1 : \text{tot}(H + \varepsilon^1) \to \mathbb{RP}^\infty$ is defined by $(w, t) \mapsto [w]$, where $[w]$ denotes the line w lies in.
- $E\Sigma_2 \times_{\Sigma_2} \mathbb{R}^2 \to B\Sigma_2$ is defined by $(z \cdot y, x) \mapsto [z] = [z \cdot y]$, which is $(w, t) \mapsto [w]$.

Therefore, the two bundles are exactly the same.

Claim 2.14. $ri_1 = s \times s$.

$\Sigma_2 \times \Sigma_2 \xrightarrow{s \times s} O(2) \times O(2) \subset O(4)$ is defined by

$$(\sigma, 0) \mapsto \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \qquad (0, \sigma) \mapsto \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix}.$$

and $\Sigma_2 \times \Sigma_2 \xrightarrow{i_1} G \xrightarrow{r} O(4)$ is defined by

$$(\sigma, 0) \mapsto (12) \mapsto \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \qquad (0, \sigma) \mapsto (34) \mapsto \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix}.$$

So $s \times s = ri_1$.

Claim 2.15. $s \otimes s = ri_2$.

$\Sigma_2 \times \Sigma_2 \xrightarrow{i_2} G \xrightarrow{r} O(4)$ is defined by

$$(\sigma, 1) \mapsto (12)(34) \mapsto \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}, (1, \sigma) \mapsto (13)(24) \mapsto \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}.$$

Since $s : \Sigma_2 \to O(2)$ is defined by $\sigma \mapsto \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, 1 \mapsto \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$,
then $s \otimes s : \Sigma_2 \otimes \Sigma_2 \to O(2) \otimes O(2)$ satisfies

$$s(\sigma \otimes 1) = \sigma \otimes s1 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \otimes \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix},$$

$$s(1 \otimes \sigma) = s1 \otimes s\sigma = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \otimes \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} = \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}.$$

So $s \otimes s = ri_2$.

Claim 2.16. $j = Bi_1$; $1 \times_{\Sigma_2} \Delta = Bi_2$.

By **Remark 2.12** i_1, i_2 are actually:

$$i_1 : \Sigma_2 \times \Sigma_2 \to (\Sigma_2 \times \Sigma_2) \rtimes \Sigma_2, \quad (\sigma, 1) \mapsto ((\sigma, 1), 1), \ (1, \sigma) \mapsto ((1, \sigma), 1),$$
$$i_2 : \Sigma_2 \times \Sigma_2 \to (\Sigma_2 \times \Sigma_2) \rtimes \Sigma_2, \quad (\sigma, 1) \mapsto ((\sigma, \sigma), 1), \ (1, \sigma) \mapsto ((1, 1), \sigma).$$

So i_1 can also be written as $\Sigma_2 \times \Sigma_2 \cong (\Sigma_2 \times \Sigma_2) \rtimes 1 \hookrightarrow (\Sigma_2 \times \Sigma_2) \rtimes \Sigma_2$. Apply functor B and use **Fact 2.4**, we obtain

$$B(\Sigma_2 \times \Sigma_2) \xrightarrow{\simeq} B(\Sigma_2 \times \Sigma_2) \times_{\{1\}} E_{\{1\}} \longrightarrow B(\Sigma_2 \times \Sigma_2) \times_{\Sigma_2} E\Sigma_2.$$

Then $Bi_1 = j$.

Consider $\Sigma_2 \times \Sigma_2 \xrightarrow{\Delta \rtimes 1} (\Sigma_2 \times \Sigma_2) \rtimes \Sigma_2$ defined by $(a,b) \mapsto ((a,a),b)$, which is a group homomorphism since $b \cdot (a,a) = (a,a)$ for any $b \in \Sigma_2$. We observe that $i_2 = \Delta \rtimes 1$.

Apply functor B and use **Fact 2.4**, we obtain

$$B\Sigma_2 \times B\Sigma_2 \xrightarrow{\Delta \times_{\Sigma_2} 1} B\Sigma_2 \times B\Sigma_2 \times_{\Sigma_2} E\Sigma_2$$

Remark 2.17. We use G being a subgroup of Σ_4 (so we have group representation) to prove $ri_1 = s \times s$ and $ri_2 = s \otimes s$. We use $G = \Sigma_2 \wr \Sigma_2$ (so $BG = E\Sigma_2 \times_{\Sigma_2} (B\Sigma_2 \times B\Sigma_2)$) in computing $j = Bi_1$, $1 \times_{\Sigma_2} \Delta = Bi_2$. □

Consider the commutative diagram:

$$\begin{array}{ccccc} tot(j^*\rho) & \longrightarrow & tot(\rho) & \longrightarrow & EO(4) \\ \downarrow{\scriptstyle j^*\rho} & & \downarrow{\scriptstyle \rho} & & \downarrow \\ B(\Sigma_2 \times \Sigma_2) & \xrightarrow[Bi_1 = j]{} & BG & \xrightarrow[Br]{} & BO(4) \end{array}$$
$$B(ri_1) = B(s \times s) = Bs \times Bs : B(\Sigma_2 \times \Sigma_2) \to BO(4)$$

So by **Claim 2.13**, $j^*\rho = (H + \varepsilon^1) \times (H + \varepsilon^1)$.

Remark 2.18. Consider a more general problem: Let $A \subset B_1$, $\pi_1 : E_1 \to B_1$, $\pi_2 : E_2 \to B_2$ be two vector bundles and $E_2 \to B_2$ has rank n. Then we have pullback:

$$\begin{array}{ccc} E_1|_A \times \epsilon^n & \longrightarrow & E_1 \times E_2 \\ \downarrow & & \downarrow{\scriptstyle \pi_1 \times \pi_2} \\ A \simeq A \times \{pt\} & \longrightarrow & B_1 \times B_2 \end{array}$$

□

By **Remark 2.18**, the bundle over $\mathbb{RP}^2$ is $(H + \varepsilon^1)|_{\mathbb{RP}^2} + \varepsilon^2 = H_2 + \varepsilon^1 + \varepsilon^2 = H_2 + \varepsilon^3$, i.e., $a^*\rho = H_2 + \varepsilon^3$.

Consider a commutative diagram:

$$\begin{array}{ccccc} tot(Bi_2^*\rho) & \longrightarrow & tot(\rho) & \longrightarrow & EO(4) \\ \downarrow{\scriptstyle Bi_2^*\rho} & & \downarrow{\scriptstyle \rho} & & \downarrow \\ B(\Sigma_2 \otimes \Sigma_2) & \xrightarrow[Bi_2]{} & BG & \xrightarrow[Br]{} & BO(4) \end{array}$$
$$Bs \otimes s : B(\Sigma_2 \otimes \Sigma_2) \to BO(4)$$

So $Bi_2^*\rho = (H+\varepsilon^1)\otimes(H+\varepsilon^1)$. Similarly as **Remark 2.18**, $b^*\rho = (H_1+\varepsilon)\otimes(H_1+\varepsilon) = H_1\otimes H_1 + H_1\otimes\varepsilon^1 + \varepsilon^1\otimes H_1 + \varepsilon^1\otimes\varepsilon^1$.

Remark 2.19. Here is a commutative diagram that helps readers understand relations.

$$\begin{array}{ccccc}
(H_1+\epsilon^1)\otimes(H_1+\epsilon^1) & \longrightarrow & (H+\epsilon^1)\otimes(H+\epsilon^1) & \longrightarrow & EO(4) \\
\downarrow & & \downarrow & & \downarrow \\
S^1\times S^1 = \mathbb{RP}^1\times\mathbb{RP}^1 & \longrightarrow & B(\Sigma_2\otimes\Sigma_2) & \xrightarrow{Bs\otimes s} & BO(4)
\end{array}$$

□

Claim 2.20. $w(H_1) = 1+x$.

Since H_1 is a line bundle, we only need to compute $w_1(H_1)$. Note that $H^*(\mathbb{RP}^1) = \mathbb{Z}_2[x]/x^2$, $H^*(\mathbb{RP}^\infty) = \mathbb{Z}_2[x]$. Then $\iota : \mathbb{RP}^1 \to \mathbb{RP}^\infty$ induces $\iota^* : H^*(\mathbb{RP}^\infty) \to H^*(\mathbb{RP}^1)$. By axiom of Stiefel-Whitney class, $w_1(H) \neq 0$, so $w_1(H) = x$. Then $w_1(\iota^* H) = \iota^*(w_1(H)) = \iota^* x = x \in H^1(\mathbb{RP}^1)$. So $w(H_1) = 1+x$.

Then we can compute:

$$w(a^*\rho) = w(H_2)w(\varepsilon^3) = w(H_2) = 1+u,$$
$$w(b^*\rho) = w(H_1\otimes H_1)w(H_1\otimes\varepsilon^1)w(\varepsilon^1\otimes H_1)w(\varepsilon^1\otimes\varepsilon^1).$$

Since they are all line bundles, we only need to compute the first Stiefel–Whitney class.

Fact 2.21. Let L_1, L_2 be real line bundles over a paracompact space B. Then $w_1(L_1\otimes L_2) = w_1(L_1) + w_1(L_2)$.

Hence,

$$\begin{aligned}
w_1(H_1\otimes H_1) &= w_1(H_1) + w_1(H_1) = x_1 + x_2, \\
w_1(H_1\otimes\varepsilon^1) &= x_1 + 0, \\
w(\varepsilon^1\otimes H_1) &= 0 + x_2, \\
w(\varepsilon^1\otimes\varepsilon^1) &= 0.
\end{aligned}$$

So

$$w(b^*\rho) = (1+x_1+x_2)(1+x_1)(1+x_2)\cdot 1 = 1 + x_1x_2$$

(using $x_1^2 = x_2^2 = 0$ and note that we are in $\mathbb{Z}_2$-coefficients).

$w_1(\xi) = w_1(c^*\rho) = w_1((a\vee b)^*\rho) = w_1(a^*\rho) + w_1(b^*\rho) = u + x_1x_2 = u + u^2$, where the third equation holds by **Remark 2.22**.

Remark 2.22. $\iota_1 : P \hookrightarrow P\vee T$ and $\iota_2 : T \hookrightarrow P\vee T$ induce projection on cohomology $\iota_1^* : H^i(P\vee T) = H^i(P)\oplus H^i(T) \to H^i(P)$ and $\iota_2^* : H^i(P\vee T) =$

$H^i(P) \oplus H^i(T) \to H^i(T)$, i.e., for any $x \in H^i(P \vee T)$, we have $x = \iota_1^* x + \iota_2^* x$. Let $f = a \vee b$. Then $f\iota_1 = a$, $f\iota_2 = b$. For any $w_i(f^*\rho) \in H^i(P \vee T)$, we have

$$\begin{aligned} w_i(f^*\rho) &= \iota_1^* w_i(f^*\rho) + \iota_2^* w_i(f^*\rho) = \iota_1^* f^* w_i(\rho) + \iota_2^* f^* w_i(\rho) \\ &= (f\iota_1)^* w_i(\rho) + (f\iota_2)^* w_i(\rho) = a^* w_i(\rho) + b^* w_i(\rho) = w_i(a^*\rho) + w_i(b^*\rho), \end{aligned}$$

for $i > 1$. □

So $w(\xi) = 1 + u + u^2$. Then

$$w(\tau\overline{X} + 7\xi) = w(\tau\overline{X})w(\xi)^7 = (1 + u + u^2)^8 = 1.$$

One can find the details of the computation showing that this manifold has Kervaire invariant one in [3].

Constructing an extended power of dimension 30 works, but can this method be used to construct an example in dimension 62? Actually, it is not easy by means of Theorem D in [3].

Let $G_k := \Sigma_2 \wr \cdots \wr \Sigma_2$ (k's Σ_2). Consider $Y_{G_k}(S^7)$ with $\dim Y = d = 2^{l+1} - 2 - 7 \cdot 2^k$. By **Remark 2.3**, $\dim Y_{G_k}(S^7) = 2^{l+1} - 2$.

$2^{l+1} - 2 = 62$ implies $l = 5$. With $\dim Y = 2^6 - 2 - 7 \cdot 2^k \geq 0$, we have the only possibilities of (k, d) are $(0, 55), (1, 48), (2, 34), (3, 6)$. All cases satisfying $d > 2$ meeting the condition of Theorem D in [3]. So given any framing of S^7, and α a stable trivialization of $\tau\overline{Y} + 7\xi$, then $K(Y_{G_k}(S^7), \alpha_{G_k}(F)) = 0$. Indeed, replacing $\alpha_{G_k}(F)$ with another framing can make the Kervaire invariant nontrivial, but it is not easy to find such a framing.

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