• 级数.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

• $\sin h + \sin 2h + \sin 3h + \cdots + \sin mh = \sum_{n=1}^{m} \sin nh$

$e^{ix} = \cos x + i\sin x$.

$e^{inh} = \cos nh + i\sin nh$. $\Rightarrow$ $\sin nh = \operatorname{Im} e^{inh}$ ← $e^{inh}$的虚部

则 $\sum_{n=1}^{m} \sin nh = \sum_{n=1}^{m} \operatorname{Im} e^{inh}$

$$= \operatorname{Im} \sum_{n=1}^{m} e^{inh}$$

$$= \operatorname{Im} \sum_{n=1}^{m} (e^{ih})^n$$ ← 用等比数列求和公式 $\sum_{q=1}^{n} a^q = \frac{a(1-q^n)}{1-q}$

$$= \operatorname{Im} \frac{e^{ih}(1-e^{ihm})}{1-e^{ih}}$$

$$= \operatorname{Im} \frac{(e^{ih} - e^{ih(m+1)})e^{-ih/2}}{(1-e^{ih})e^{-ih/2}}$$

$$= \operatorname{Im} \frac{e^{\frac{ih}{2}} - e^{ih(m+\frac{1}{2})}}{e^{-\frac{ih}{2}} - e^{\frac{ih}{2}}}$$

$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$$= \operatorname{Im} \frac{\cos\frac{h}{2} + i\sin\frac{h}{2} - \left(\cos\left(h(m+\frac{1}{2})\right) + i\sin h(m+\frac{1}{2})\right)}{-2i\sin\theta}$$

$$= \operatorname{Im} \frac{i\cos\frac{h}{2} - \sin\frac{h}{2} - i\cos h(m+\frac{1}{2}) + \sin\left(h(m+\frac{1}{2})\right)}{2\sin\theta}$$

$$= \frac{\cos\frac{h}{2} - \cos(h(m+\frac{1}{2}))}{2\sin\theta}$$

- $\int f(x)\,dx$：不定积分　　$\int_a^b f(x)\,dx$：定积分

积分上下限　　被积函数.

- $\int_a^b f(t)\,dx = ?$　　$\left(\int_a^b f(t)\,dx = f(t)\int_a^b dx = f(t)(b-a).\right)$

- $\int f(x)\,dx$ 不定积分要加常数C

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- 积分的性质

① 积分的线性性

Rmk：什么是线性性？就是常数可提到外面，加法可拆的性质。
这里罗列一些有线性性的算子.

$\lim$，$\frac{d}{dx}$，$\int$，线性变换.

△ $\int a f(x)\,dx = a\int f(x)\,dx$　（$\int$ 也可以是定积分 $\int_s^t$）

Rmk：只要是常数就可提. $\int f(t)\,dx = f(t)\int dx$，因为 $f(t)$ 相对于 $x$ 而言是常数.

△ $\int f(x)+g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$.

② $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

③ $\int_a^a f(x)\,dx = 0$

④ $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

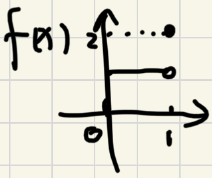
⑤ 在 $[a,b]$ 上, $f(x) \geq g(x)$ 则 $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

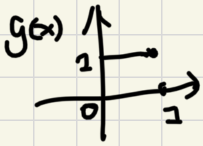
特别地, 在 $[a,b]$ 上 $f(x) \geq 0$ 则 $\int_a^b f(x)\,dx \geq 0$

Rmk: 0是一个特殊的函数.

⑥ 粗略地说, $f(x)$ 和 $g(x)$ 只在若干个点处不同, 仍有

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$$




则 $\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx = 1$

⑦ 不是所有函数都可积的

⑧ 若 $F'(x) = f(x)$, 则 $F(x)$ 是 $f(x)$ 的一个原函数. $\int f(x)\,dx = F(x) + C$.

如果 $f(x)$ 有原函数 则 $\int f(x)\,dx$ 可积; 如果 $\int f(x)\,dx$ 可积, 不一定能求出原函数(写不出表达式, 只知道它可积)

⑨ $$\int_a^b f(x)\,dx = F(b) - F(a)$$

⑩ $F'(x) = f(x)$

$$\int_{a(x)}^{b(x)} f(t)\,dt = F(b(x)) - F(a(x))$$

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = \frac{d}{dx}\Big(F(b(x)) - F(a(x))\Big)$$

$$= f(b(x))\,b'(x) - f(a(x))\,a'(x)$$

例 5.4 T7　$f'(x)>0$.　$f(1)=0$

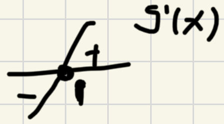
令 $g(x)=\int_0^x f(t)\,dt$.

△ $\frac{d}{dx}g(x)=f(x)$，因此 $g(x)$ 可导.

因为 $g(x)$ 可导，故 $g(x)$ 一定连续.

△ $\frac{d}{dx}g(x)\Big|_{x=1}=f(1)=0$ 故 $g(x)$ 在 $x=1$ 有水平切线. $g''(x)=f'(x)>0$.

△ $g'(x)\Big|_{x=1}=f(1)=0$.




$x=1$ 是 $g(x)$ 的 local min.

△ $g''(1)>0$ 故不是拐点.

△ $g'(1)=f(1)=0$　$g'(x)$ 在 $x=1$ 时穿过 x 轴.

例. Find upper and lower bound for $\int_0^1 \frac{1}{1+x^2}\,dx$

$\frac{1}{1+x^2}$ 在 $[0,1]$ 上单调递减.

故 $\frac{1}{2}\le\frac{1}{1+x^2}\le 1$.

$$\underbrace{\frac{1}{2}}_{\text{lower bound}}=\int_0^1\frac{1}{2}\,dx\le\int_0^1\frac{1}{1+x^2}\,dx\le\int_0^1 1\,dx=\underbrace{1}_{\text{upper bound}}$$

例:

$$h(s)=\int_s^{s^2}\sqrt{1+x^2}\,dx$$

令 $F(x)=\int\sqrt{1+x^2}$

$$\frac{d}{ds}h(s)=\frac{d}{ds}\left(F(s^2)-F(s)\right)$$
$$=\sqrt{1+s^4}\,2s-\sqrt{1+s^2}$$

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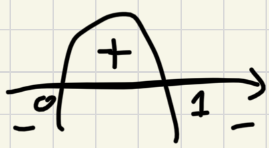
• 积分的几何含义：图像下的面积（<u>有正负</u>）

What values of a and b maximize the value of $\int_a^b (x-x^2)\,dx$ ?




把正面积全积起来.

所以 $a=0$, $b=1$ 时值最大.

- 求导公式(也是积分公式)

$(\tan x)' = \frac{1}{\cos^2 x} = \sec^2 x$

$(\cot x)' = -\frac{1}{\sin^2 x}$

$(\arctan x)' = \frac{1}{1+x^2}$

$(\text{arccot}\, x)' = -\frac{1}{1+x^2}$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

$(\sinh x)' = \cosh x$

$(\cosh x)' = \sinh x$

$(\text{arcsinh}\, x)' = \frac{1}{\sqrt{x^2+1}}$

$(\text{arccosh}\, x)' = \frac{1}{\sqrt{x^2-1}}$

$\int \frac{1}{x} dx = \ln|x| + C$ $\quad \int \sin x \, dx = -\cos x$

$\int x^m dx = \frac{x^{m+1}}{m+1} + C$ $\quad \int \cos x \, dx = \sin x$

* $\sec x = \frac{1}{\cos x}$ $\quad \csc x = \frac{1}{\sin x}$ $\quad \sinh x = \frac{e^x - e^{-x}}{2}$ $\quad \cosh x = \frac{e^x + e^{-x}}{2}$

$\sinh 2x = 2\sinh x \cosh x$ $\quad \cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh^2 x - \sinh^2 x = 1$

- 6.7T52

$$\int 2 + \tan^2\theta \, d\theta$$

$$= \int 2 d\theta + \int \tan^2\theta \, d\theta$$

$$= 2\theta + \int \frac{\sin^2\theta}{\cos^2\theta} d\theta$$

$$= 2\theta + \int \frac{1-\cos^2\theta}{\cos^2\theta} d\theta$$

$$= 2\theta + \int \frac{1}{\cos^2\theta} d\theta - \int 1 \, d\theta$$

$$= 2\theta + \tan\theta - \theta + C$$

$$= \theta + \tan\theta + C$$

5.4

T16

$$\int_0^{\pi/6} (\sec x + \tan x)^2 dx$$

$$= \int_0^{\pi/6} \sec^2 x + \tan^2 x + 2\sec x \tan x \, dx$$

$$= \underbrace{\int_0^{\pi/6} \sec^2 x \, dx}_{\mathrm{I}} + \underbrace{\int_0^{\pi/6} \tan^2 x \, dx}_{\mathrm{II}} + \underbrace{2\int \sec x \tan x \, dx}_{\mathrm{III}}.$$

$$\mathrm{I} = \int_0^{\pi/6} \sec^2 x \, dx$$

$$= \tan x \Big|_0^{\pi/6}$$

$\text{II} = \int_0^{\pi/6} \tan^2 dx$

$= \int_0^{\pi/6} \frac{1-\cos^2 x}{\cos^2 x} dx$

$= \int_0^{\pi/6} \frac{1}{\cos^2 x} - 1 dx$

$= \tan x \Big|_0^{\pi/6} - x \Big|_0^{\pi/6}$

$\text{III} = 2 \int_0^{\pi/6} \sec x \tan x \, dx$

$= 2 \int_0^{\pi/6} \frac{\sin x}{\cos^2 x} dx$

$= -2 \int_0^{\pi/6} \frac{1}{\cos^2 x} d\cos x$

$= 2 \frac{1}{\cos x} \Big|_0^{\pi/6}$

原式 $= 2 \tan x \Big|_0^{\pi/6} - x \Big|_0^{\pi/6} + 2 \frac{1}{\cos x} \Big|_0^{\pi/6}$

$= 2 (\tan \pi/6) - \pi/6 + 2 \left( \frac{1}{\cos \pi/6} - 1 \right)$

$= 2\sqrt{3} - \pi/6 - 2$

T31

$\int_2^5 \frac{x\,dx}{\sqrt{1+x^2}}$

$= \int_2^5 \frac{\frac{1}{2} \cdot 2x\,dx}{\sqrt{1+x^2}}$

$= \frac{1}{2} \int_2^5 \frac{dx^2}{\sqrt{1+x^2}}$

$= \frac{1}{2} \int_2^5 \frac{d(1+x^2)}{\sqrt{1+x^2}}$

$= \frac{1}{2} \cdot \frac{(1+x^2)^{1/2}}{1/2} \Big|_2^5$

$= \sqrt{1+x^2} \Big|_2^5 = \sqrt{26} - \sqrt{5}$

T36

a) $\int_0^{\tan\theta} \sec^2 y \, dy$

$= \tan y \Big|_0^{\tan\theta}$

$= \tan \tan\theta$

b) $\frac{d}{d\theta} \tan \tan\theta$

$= \frac{1}{\cos^2 \tan\theta} \frac{1}{\cos^2\theta}$

T46.

$y = \int_{\tan x}^0 \frac{dt}{1+t^2}$

$= \arctan t \Big|_{\tan x}^0$

$= 0 - x = -x$

$\frac{d}{dx} y = \frac{d}{dx}(-x) = -1$

T2. $f(x)$在$[a,b]$上恒正且连续.

问 $\int_a^x f(t)dt + \int_b^x f(t)dt=0$ 在$(a,b)$上有几个根?

$$F(x)=\int_a^x f(t)dt + \int_b^x f(t)dt$$

在$[a,b]$上连续、可导.

$$F(a)=\int_a^a f(t)dt + \int_b^a f(t)dt$$

$$= -\int_a^b f(t)dt < 0$$

$$F(b)=\int_a^b f(t)dt + \int_b^b f(t)dt$$

$$= \int_a^b f(t)dt > 0.$$

故 $F(x)=0$ 在$(a,b)$内一定有解.

$F'(x)=2f(x)>0$.

故$F(x)$在$(a,b)$上单调递增.

因此$F(x)$在$(a,b)$上只有一解.

选 B.

T10

$$\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{1+\cos\frac{\pi}{n}}+\sqrt{1+\cos\frac{2\pi}{n}}+\cdots+\sqrt{1+\cos\frac{n\pi}{n}}\right)$$

$$= \int_0^1 \sqrt{1+\cos\pi x}\,dx$$

$$= \int_0^1 \frac{\sqrt{1+\cos\pi x}\sqrt{1-\cos\pi x}}{\sqrt{1-\cos\pi x}}dx$$

$$= \int_0^1 \frac{\sqrt{1-\cos^2\pi x}}{\sqrt{1-\cos\pi x}}dx$$

$x\in[0,1]$时 $\sin\pi x>0$

$$= \int_0^1 \frac{\sin\pi x}{\sqrt{1-\cos\pi x}}dx$$

$$= \frac{1}{\pi}\int_0^1 \frac{\sin\pi x}{\sqrt{1-\cos\pi x}}\,d\pi x$$

$$= -\frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{1-\cos\pi x}}\,d\cos\pi x$$

$$= \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{1-\cos\pi x}}\,d(1-\cos\pi x)$$

$$= \left.\frac{(1-\cos\pi x)^{1/2}}{\frac{1}{2}\pi}\right|_0^1 = \frac{2}{\pi}\sqrt{1-\cos\pi x}$$

$$= \frac{2\sqrt{2}}{\pi}$$

T20 $\int_0^{x^2-1} f(t)\,dt = x-1$. 求 $f(7)$

令 $F(t)=\int f(t)\,dt$

$\int_0^{x^2-1} f(t)\,dt = F(x^2-1)-F(0) = x-1$.

$\Rightarrow F(x^2-1) = x-1+F(0)$.

$\frac{d}{dx}F(x^2-1)$

$= f(x^2-1)\,2x$

$= \frac{d}{dx}(x-1+F(0))$

$= 1$

故 $f(x^2-1)\,2x = 1$.

令 $x=\sqrt{8}$ 得 $f(7)\,2\sqrt{8} = 1$

$\Rightarrow f(7) = \frac{1}{2\sqrt{8}} = \frac{1}{4\sqrt{2}}$

- 傅里叶级数（早鸟版）（不考）

在线代中，任一向量可以表示成基的线性组合，如 $v=\sum_{n=1}^{m} k_n b_n$.

在函数空间中，类似地，也有它的"基"：

$1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx, \dots \dots$

因此工程上遇见的函数可以表示成：

$$f(x) = \sum_{n=0}^{\infty} a_n\cos nx + b_n \sin nx$$

$$= a_0\cos 0x + a_0 \sin 0x + \sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx$$

$$:= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \qquad (*)$$

就是常数，记作 $\frac{a_0}{2}$ 是一种 convention

这组基有一个无比好的性质：正交性

$$\int_{-\pi}^{\pi}\cos nx\,dx=\begin{cases}2\pi & n=0\\ 0 & n\neq 0\end{cases} \qquad \int_{-\pi}^{\pi}\sin nx\,dx=0$$

$$\int_{-\pi}^{\pi}\cos mx\cos nx\,dx=\int_{-\pi}^{\pi}\frac{1}{2}\left[\cos(m-n)x+\cos(m+n)x\right]dx=\pi\delta_{mn}$$

$$\int_{-\pi}^{\pi}\sin mx\sin nx\,dx=\int_{-\pi}^{\pi}\frac{1}{2}\left[\cos(m-n)x+\cos(m+n)x\right]dx=\pi\delta_{mn}$$

$$\int_{-\pi}^{\pi}\cos mx\sin nx\,dx=\int_{-\pi}^{\pi}\frac{1}{2}\left[\sin(m+n)x-\sin(m-n)x\right]dx=0$$

对(*)两边同乘 $\cos nx$ 并计算 $[-\pi,\pi]$ 上的积分

$$\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$=\frac{a_0}{2}\int_{-\pi}^{\pi}\cos nx\,dx+\sum_{k=1}^{\infty}\Big(\underbrace{a_k\int_{-\pi}^{\pi}\cos kx\cos nx\,dx}_{\text{只留下}k=n\text{的项}}+\underbrace{b_k\int_{-\pi}^{\pi}\sin kx\cos nx\,dx}_{=0}\Big)$$

由正交性，当 $n=0$

$$\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{a_0}{2}\cdot 2\pi=a_0\pi$$

当 $n\neq 0$

$$\int_{-\pi}^{\pi}f(x)\cos nx\,dx=a_n\pi$$

$$\Rightarrow \int_{-\pi}^{\pi}f(x)\cos nx=\pi a_n.$$

同理可得 $\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\pi b_n.$

$$\begin{cases}a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx\\ b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx\end{cases}$$

算积分：

1. $\boxed{\text{分部积分法}}$

$$\int \ln x \, dx$$

$$= x\ln x - \int x \frac{1}{x} \, dx$$

$$= x\ln x - x + C$$

2. $$\int \sec x \, dx$$

$$= \int \frac{1}{\cos x} \, dx$$

$$= \int \frac{\cos x}{\cos^2 x} \, dx$$

$$= \int \frac{1}{\cos^2 x} \, d\sin x$$

$$= \int \frac{1}{1-\sin^2 x} \, d\sin x$$

$$= \int \frac{1}{(1-\sin x)(1+\sin x)} \, d\sin x$$

$$= \frac{1}{2}\int \frac{1}{1+\sin x} + \frac{1}{1-\sin x} \, d\sin x$$

$$= \frac{1}{2}\Big[\ln|1+\sin x| - \ln|1-\sin x|\Big] + C = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

4. $$\int_0^{2\pi} \sqrt{1-\sin 2x} \, dx$$

$$= \int_0^{2\pi} \sqrt{\sin^2 x + \cos^2 x - \sin 2x} \, dx$$

$$= \int_0^{2\pi} |\sin x - \cos x| \, dx$$

$$= \int_0^{\pi/4} \cos x - \sin x \, dx + \int_{\pi/4}^{5\pi/4} \sin x - \cos x \, dx$$

$$+ \int_{5\pi/4}^{2\pi} \cos x - \sin x \, dx$$

$$= \cdots\cdots$$ (略)

5. $\int \sin^2 x \cos^3 x \, dx$

$= \int \sin^2 x \cos^2 x \, d\sin x$

$= \int \sin^2 x (1-\sin^2 x) \, d\sin x$

$= \int \sin^2 x - \sin^4 x \, d\sin x$

$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

6. $\int \frac{1+x+x^2}{x(1+x^2)} dx$

$= \int \frac{1+x^2+x}{x(1+x^2)} dx$

$= \int \frac{1}{x} dx + \int \frac{1}{1+x^2} dx$

$= \ln|x| + \arctan x + C$

7. $\int \frac{x^4}{1+x^2} dx$

$= \int \frac{x^4-1+1}{1+x^2} dx$

$= \int x^2-1 \, dx + \int \frac{1}{1+x^2} dx$

$= \frac{x^3}{3} - x + \arctan x + C$

8. $\int \frac{dx}{x^2(1+x^2)} dx$

$= \int \frac{dx}{x^2} - \int \frac{dx}{1+x^2}$

$= -x^{-1} - \arctan x + C$

9. $\int \frac{dx}{\sin^2 x \cos^2 x}$

$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$ 凑齐次

$= \int \frac{1}{\cos^2 x} dx + \int \frac{1}{\sin^2 x} dx$

$= \tan x - \cot x + C$

10. $\int (ax+b)^m dx$

$= \frac{1}{a} \int (ax+b)^m d(ax+b)$

$m = -1$

原式 $= \frac{1}{a} \ln|ax+b|$

$m \neq -1$

原式 $= \frac{(ax+b)^{m+1}}{a(m+1)}$

11. $\int \frac{dx}{a^2+x^2}$

$= \frac{1}{a^2} \int \frac{dx}{1+\frac{x^2}{a^2}} = \frac{1}{a} \int \frac{d\frac{x}{a}}{1+\frac{x^2}{a^2}}$

$= \frac{1}{a} \arctan \frac{x}{a} + C$

12. $\int \tan x \, dx$

$= \int \frac{\sin x}{\cos x} dx$

$= -\int \frac{1}{\cos x} d\cos x$

$= -\ln|\cos x|$

13. $\int \cot x \, dx$

$= \int \frac{\cos x}{\sin x} dx$

$= \int \frac{d\sin x}{\sin x}$

$= \ln|\sin x| + C$